

Equal circles in a circle: optimal radii for 15, 16, 17, 18, and 20 circles

Abstract

We prove the least container radii for fifteen, sixteen, seventeen, eighteen and twenty unit disks with disjoint interiors. They are $1 + \sqrt{6 + 2/\sqrt{5} + 4\sqrt{1 + 2/\sqrt{5}}}$ for fifteen, $1 + \sqrt{2} + \sqrt{6}$ for eighteen, and the three exactly specified algebraic radii of the classical constructions for the other counts. The optimal packings for fifteen, sixteen and seventeen are unique up to isometry and relabeling. For eighteen we give a continuous family of optimal packings. Every optimal twenty-disk packing consists of a specified nineteen-center configuration and one center in an explicitly defined feasible region. The lower bounds use exhaustive subdivision of radial intervals and cyclic orders, certified with exact arithmetic. An odd-cycle inequality or an explicit local-rigidity bound completes each proof.

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1 Introduction

1.1 The problem and the main theorem

For a positive integer N , let R_N be the least radius of a closed disk containing N closed unit disks with pairwise disjoint interiors. Tangencies are allowed. We fix the container center at the origin O . A unit disk centered at a vector $\mathbf{x} \in \mathbb{R}^2$ lies in a container of radius R exactly when $\|\mathbf{x}\| \leq R - 1$. Two unit disks have disjoint interiors exactly when their centers are at distance at least 2. Thus, writing $b = R - 1$, the conditions are

$$\|\mathbf{x}_i\| \leq b \quad (1 \leq i \leq N), \quad \|\mathbf{x}_i - \mathbf{x}_j\| \geq 2 \quad (1 \leq i < j \leq N). \quad (1.1)$$

We call $\{\mathbf{x}_1, \dots, \mathbf{x}_N\}$ a *feasible center set* in the disk of radius b . The minimum exists: a sequence approaching the infimum has all coordinates in a fixed bounded disk, and a convergent subsequence preserves the closed inequalities (1.1).

In each case section, point labels such as $\mathbf{A}$ and $\mathbf{Q}_j$ refer to that section's construction. The case number remains on R_N , b_N and $\mathcal{P}_N$ when the five results are compared. The finite sets $\mathcal{P}_N$ describe the classical constructions: Pirl's for fifteen circles, Goldberg's for sixteen and twenty, and Reis's for seventeen, as attributed in [8, Section 2] and discussed below. The case sections give exact coordinate formulas. Table 1 gives approximate values of the optimal radii.

Theorem 1.1. *Let b_{16} be the least root of (8.5) in the rational interval of Appendix A.1. Let b_{17} and b_{20} be the unique roots of (9.6) and (11.3) in the intervals (9.1) and (11.1), respectively. The coordinate constructions in Sections 8.1, 9.1 and 11.1 define these equations. The least container radii are*

$$\begin{aligned} R_{15} &= 1 + \sqrt{6 + 2/\sqrt{5} + 4\sqrt{1 + 2/\sqrt{5}}}, \\ R_{16} &= 1 + b_{16}, \quad R_{17} = 1 + b_{17}, \\ R_{18} &= 1 + \sqrt{2} + \sqrt{6}, \quad R_{20} = 1 + b_{20}. \end{aligned}$$

For $N \in \{15, 16, 17\}$, every optimal packing is congruent to the packing with center set $\mathcal{P}_N$, up to relabeling. Congruence permits translations, rotations and reflections of the packing together with its container.

Put $b_{18} = \sqrt{2} + \sqrt{6}$ and $\mathbf{e}(\theta) = (\cos \theta, \sin \theta)$. For every $\theta \in [\pi/6, \pi/3]$, the center set

$$\begin{aligned} & \{O\} \cup \{2\mathbf{e}(\pi/12 + j\pi/3) : j \in \{0, \dots, 5\}\} \\ & \cup \{b_{18}\mathbf{e}(j\pi/6) : j \in \{0, 3, 4, 5, 6, 7, 8, 9, 10, 11\}\} \cup \{b_{18}\mathbf{e}(\theta)\} \end{aligned} \quad (1.2)$$

gives an optimal eighteen-disk packing. Optimal packings are not unique up to congruence: the members with $\theta = \pi/6$ and $\theta = \pi/4$ are not congruent.

Let $\mathcal{C}$ be the nineteen-point set defined in Section 11.1, and put

$$\mathcal{A} = \{\mathbf{z} \in \mathbb{R}^2 : \|\mathbf{z}\| \leq b_{20}, \|\mathbf{z} - \mathbf{p}\| \geq 2 \text{ for every } \mathbf{p} \in \mathcal{C}\}. \quad (1.3)$$

With the container centered at O , the optimal twenty-disk center sets are precisely the orthogonal images of $\mathcal{C} \cup \{\mathbf{z}\}$ with $\mathbf{z} \in \mathcal{A}$. In particular, every position with $\|\mathbf{z}\| \leq 17/100$ belongs to $\mathcal{A}$ and is permitted for the twentieth center.

The eighteen-circle assertion specifies a family of optimizers; a complete classification of that equality case remains outside the result. The twenty-circle assertion gives both a necessary fixed subset and the full condition on the remaining center.

N	Exact construction	R_N , nearest 10^{-9}
15	Pentagon with outward squares	4.521356965
16	Section 8.1	4.615425595
17	Section 9.1	4.792033748
18	Hexagon and dodecagon	4.863703305
20	Section 11.1	5.122320737

Table 1: Optimal container radii. The last column is rounded to the nearest multiple of 10^{-9} . All bounds in the proofs have exact rational endpoints.

1.2 Earlier constructions and proofs

Graham, Lubachevsky, Nurmela and Östergård give a historical account of the constructions in [8, Section 2]. They describe Kravitz’s examples of 1967 [2], Graham’s optimality results through seven circles in 1968 [3], and Pirl’s 1969 results through ten circles [4]. Their account attributes the fifteen-circle construction to Pirl [4], the sixteen- and twenty-circle constructions to Goldberg in 1971 [5], and the improved seventeen-circle construction to Reis in 1975 [6].

The hexagon–dodecagon configuration for eighteen circles has an earlier occurrence. Adding its center gives the nineteen-point configuration of Bateman and Erdős [1, p. 306], after doubling their lengths. Figure 3 of [8] displays several eighteen-circle packings at the same radius. Theorem 1.1 of the present paper proves that radius is least and verifies the family (1.2).

Melissen’s eleven-circle result appeared in 1994 [9]. Fodor’s nineteen-, twelve- and thirteen-circle results appeared in 1999, 2000 and 2003, respectively [10, 11, 12]. Ekanayake and LaFountain’s fourteen-circle proof appeared in 2024; they describe it as independent and cite Fodor’s 2003 fourteen-circle result [7] [13, Introduction and Theorem 4.1]. Their introduction records proofs only for counts through fourteen and for nineteen. Thus the five cases $N =$

15, 16, 17, 18, 20 treated here were open in that 2024 account. We prove matching lower bounds for all five.

1.3 What is new and the proof method

To describe the geometric method, we distinguish two kinds of contact. A *pair contact* means that two unit disks touch, so their centers are at distance exactly 2. We also call this a contact between the two centers. A *container contact* means that a unit disk touches the container, so its center has norm b . A contact chain is a sequence in which successive centers are in pair contact.

What is new. We prove lower bounds matching the five constructions and establish the equality conclusions of Theorem 1.1. The decisive computational step subdivides radial intervals for every permitted cyclic order and certifies all exclusions by rational inequalities. The retained ranges are small enough for a final exact argument: an odd-cycle inequality for fifteen circles and quantitative local-rigidity estimates for the other cases. The radial and angular restrictions have precedents in circle-packing proofs. The positive equilibrium stress used in the final step comes from general rigidity theory [14, Section 2.3]; Section 5.4 defines it and proves the quantitative criterion used here.

Bateman and Erdős use annular restrictions and excluded angular sectors [1, Section 3]. Fodor’s thirteen-circle proof combines radial case divisions, angle sums and diameter bounds [12, Sections 1–2]. Ekanayake and LaFountain use contact chains to reduce fourteen circles to a one-parameter family, then optimize that parameter [13, Theorems 3.1 and 4.1]. Here the global computation uses distances and cyclic orders. Only after the points lie close to a reference packing do its contacts force exact positions. Section 12 gives the fuller comparison.

Section 2 proves the radial projection and angular bounds. Section 3 gives shared geometric restrictions between consecutive outer directions. Section 4 specifies finite exclusion tests and proves their coverage. Section 5 proves explicit conditions under which nearby feasible points must coincide with a reference set. Section 6 records coordinate formulas for constructing prescribed contacts. Sections 7–11 apply these tools to the five counts. The appendices give arithmetic proofs, computation records and certificate data. The runnable verification evidence is supplied as separate supplementary material. Appendix H identifies its contents, maps each computational proposition to its implementation, and explains the completed verification record.

Paragraphs headed *Proof strategy* preview an argument; the justifications follow them. Each proof develops its argument in order, so the preview can also be read after the proof.

Figure 1 shows one example for each of the five counts. The marked motion in the eighteen-circle panel follows the family (1.2); the free center in the twenty-circle panel varies within (1.3).

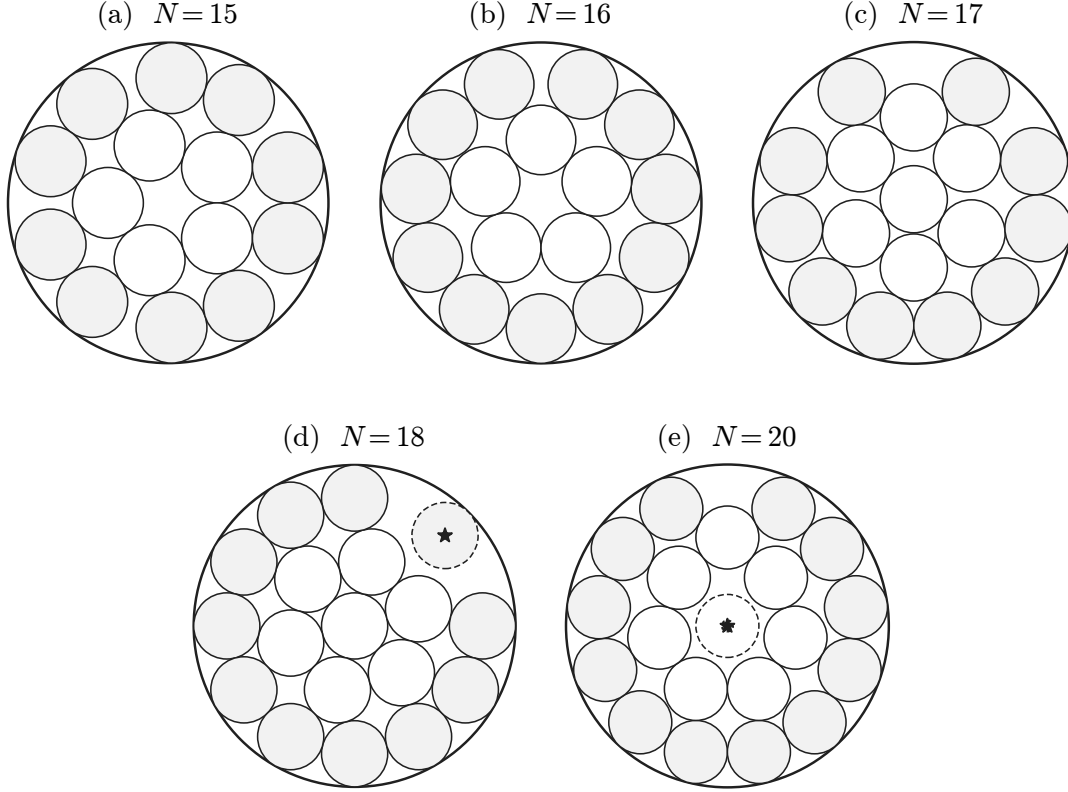


Figure 1: One optimal packing for each count: (a) fifteen, (b) sixteen, (c) seventeen, (d) eighteen and (e) twenty unit disks. In (d) and (e), a star marks the variable center and its unit circle has a dashed outline. The motion in (d) is shown separately in Figure 12; the small dotted circle in (e) marks a disk of permitted positions for the free center.

2 Radial projection and angular bounds

Proof strategy. An explicit packing gives an upper bound for the container radius. For a matching lower bound, we consider every feasible center set at that radius. We first project some centers to the boundary while preserving separation. We then use angular separation to bound the numbers and radii of the remaining centers. We also show how a contact after projection can identify a center that was already on the boundary before projection. Such a center excludes every smaller container.

2.1 Distances, directions and radial intervals

Throughout the geometric arguments, $b > 2$ denotes a fixed radius available to the centers. The *radial distance* of a center $\mathbf{x}_i$ is $r_i = \|\mathbf{x}_i\|$, its distance from O . In polar coordinates, $\mathbf{x}_i = r_i \mathbf{e}(\theta_i)$, where $\mathbf{e}(\theta) = (\cos \theta, \sin \theta)$ as in Theorem 1.1. We use “radius of a center” to mean this polar radius; each packed disk still has radius 1. A radial interval is an interval of possible values of r_i . All angles are in radians. A direction is an angle modulo 2π ; when $\mathbf{x}_i = O$, we may assign any direction to it.

We will repeatedly compare distances while keeping directions fixed. For nonnegative radial variables r, s and an angle θ , the cosine rule gives the squared distance

$$D(r, s; \theta) := \|r\mathbf{e}(0) - s\mathbf{e}(\theta)\|^2 = r^2 + s^2 - 2rs \cos \theta. \quad (2.1)$$

For fixed r, s , this function is nondecreasing in θ on $[0, \pi]$, and strictly increasing there when $r, s > 0$. In particular, two points on the circle of radius b have distance $2b \sin(\theta/2)$ when their smaller angular separation is θ . Two centers at distance at least 2 also satisfy the radial inequality

$$2 \leq \|\mathbf{x}_i - \mathbf{x}_j\| \leq r_i + r_j. \quad (2.2)$$

For example, if $r_j \leq r_j^+$, then $r_i \geq 2 - r_j \geq 2 - r_j^+$. Thus an upper bound for one radial distance gives a lower bound for every other radial distance.

The squared distance has another useful property: over a rectangle of possible radii, its maximum occurs at a corner. We prove this once so that later interval tests can use it directly.

Lemma 2.1 (Maximum at a corner). *Let $0 \leq r^- \leq r^+$ and $0 \leq s^- \leq s^+$ be interval endpoints. For every fixed real angle θ and all $(r, s) \in [r^-, r^+] \times [s^-, s^+]$,*

$$D(r, s; \theta) \leq \max_{\substack{r' \in \{r^-, r^+\} \\ s' \in \{s^-, s^+\}}} D(r', s'; \theta). \quad (2.3)$$

Proof. For fixed s , the function in (2.1) is a quadratic in r with leading coefficient 1. If $r = (1 - t)r^- + tr^+$ with $0 \leq t \leq 1$, direct expansion gives

$$D(r, s; \theta) = (1 - t)D(r^-, s; \theta) + tD(r^+, s; \theta) - t(1 - t)(r^+ - r^-)^2.$$

It is therefore at most the larger endpoint value. Apply the same argument in s to each endpoint value. The resulting four-corner maximum proves (2.3), also when an interval consists of a single point. $\square$

2.2 Radial projection preserves separation

Moving a center away from O can decrease its distance to another center. We need a threshold above which projection to radius b does not decrease any distance between centers. The following lemma identifies that threshold as

$$c(b) := b - \frac{4}{b}. \quad (2.4)$$

The argument uses only the original separation and the common upper bound b on all radii.

Lemma 2.2 (Radial projection). *Let $\mathcal{X} \subset \mathbb{R}^2$ be a finite feasible center set in the disk of radius $b > 2$. If $\mathbf{x} = r\mathbf{e}(\theta)$ belongs to $\mathcal{X}$ and $c(b) \leq r \leq b$, replacing $\mathbf{x}$ by $b\mathbf{e}(\theta)$ does not decrease its distance to any other center.*

Proof. Take another center of radius $s \leq b$, and let α_{sep} be the smaller angle between its direction and that of $\mathbf{x}$. The original centers have distance at least 2, so $D(r, s; \alpha_{\text{sep}}) \geq 4$. Rearranging this inequality and dividing by $r \geq c(b) > 0$ gives

$$2s \cos \alpha_{\text{sep}} \leq r + \frac{s^2 - 4}{r}.$$

Subtracting squared distances and using this inequality yields

$$\begin{aligned} D(b, s; \alpha_{\text{sep}}) - D(r, s; \alpha_{\text{sep}}) &= (b - r)(b + r - 2s \cos \alpha_{\text{sep}}) \\ &\geq (b - r) \left(b - \frac{s^2 - 4}{r} \right) \geq 0. \end{aligned} \quad (2.5)$$

The last inequality holds because $b - r \geq 0$ and $br \geq b^2 - 4 \geq s^2 - 4$. This proves the asserted comparison for every other center, including a center at O . $\square$

Apply Lemma 2.2 successively to all centers with radius at least $c(b)$. At each step the remaining radii are still at most b , so the lemma continues to apply. Denote the resulting set by $\hat{\mathcal{X}}$ and its centers by $\hat{\mathbf{x}}_i$. We call its centers of radius b *outer centers*, and its other centers *inner centers*. If their numbers are m and k , then $m + k = N$, and every inner radius is strictly less than $c(b)$. The strict inequality includes the threshold case, which is projected to the boundary.

Figure 2 illustrates the projection and its threshold. The mathematical comparison is (2.5); thus the figure also applies when other centers have already been projected.

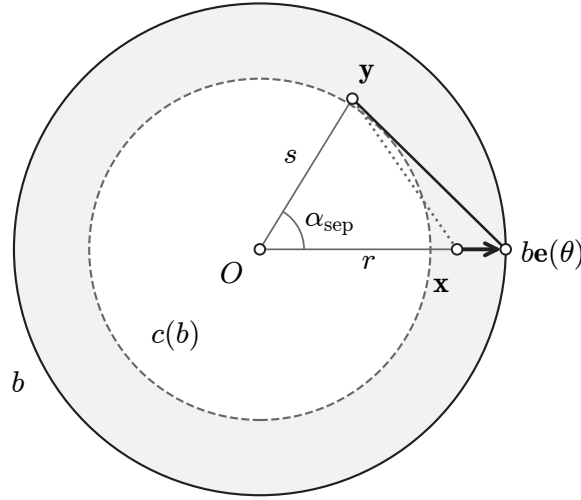


Figure 2: Radial projection of a center to the circle of radius b . The arrow joins $\mathbf{x} = r\mathbf{e}(\theta)$ to $b\mathbf{e}(\theta)$; $\mathbf{y}$ is the other center, and α_{sep} is their angle at O . The dashed circle has radius $c(b) = b - 4/b$. Lemma 2.2 proves that a center on or outside the dashed circle can be projected without decreasing any distance.

2.3 Contacts recover original boundary centers

A contact between centers means that their distance is exactly 2. To prove a lower bound after projection, we need a contact to give information about the original set. Strictness in the preceding calculation supplies that information.

Lemma 2.3 (Original boundary centers). *For the projection of a feasible center set in radius $b > 2$, an outer–inner contact implies that its outer center originally had radius b . An outer–outer contact implies that at least one of its two centers originally had radius b .*

Proof. Suppose a center moves from radius $r < b$ while another center still has radius $s < b$. The projected center satisfies $r \geq c(b)$, so

$$br \geq b^2 - 4 > s^2 - 4.$$

Both factors on the right of (2.5) are then positive. Their distance becomes strictly greater than 2, and subsequent projections cannot decrease it by Lemma 2.2. They cannot therefore be in contact in the final set.

An inner center never moves and always has radius less than b . Thus its outer contact partner cannot have moved. If both partners of an outer–outer contact had moved, the other partner would still have radius less than b at the earlier move. The same strict comparison rules this out. These two conclusions prove the lemma. $\square$

Consequently, either kind of contact in the projected set excludes a packing in every smaller center disk. Indeed, a packing in radius $b' < b$ would also be feasible in radius b , but none of its original centers could have radius b . This observation will supply the lower bound once such a contact has been forced.

Lemma 2.4 (Two possible original radii). *Suppose an outer center in a projected feasible set is in contact with a center that originally had radius b . Its own original radius is either $c(b)$ or b .*

Proof. Let θ be the smaller angle between the two directions. Their projected radii are b , and their contact gives $D(b, b; \theta) = 4$, hence $\cos \theta = 1 - 2/b^2$. The unknown original radius r lies in $[c(b), b]$, and projection preserves its direction. Substitution gives

$$D(r, b; \theta) - 4 = r^2 - 2(b - 2/b)r + b^2 - 4 = (r - b)(r - c(b)).$$

Separation makes this product nonnegative. On $[c(b), b]$ the factors have opposite signs unless one is zero, so $r = c(b)$ or $r = b$. $\square$

2.4 Required angular separation

We call a set of centers *separated* when every pair has distance at least 2. Order their directions around O ; the gaps between consecutive directions sum to 2π . We can exclude proposed radii when their required angular gaps exceed that length. To express the required angle uniformly, define, for $r, s \geq 0$,

$$g(r, s) = \begin{cases} 0, & |r - s| \geq 2, \\ \pi, & |r - s| < 2 \text{ and } r + s \leq 2, \\ \arccos \frac{r^2 + s^2 - 4}{2rs}, & |r - s| < 2 < r + s. \end{cases} \quad (2.6)$$

In the last case $r, s > 0$, and the strict triangle inequalities place the quotient in $(-1, 1)$. Whenever two centers with these radii can have distance at least 2, their smaller angular separation is at least $g(r, s)$. This statement follows from (2.1) in the last case and from radial separation in the first. If $r + s = 2$ with positive radii, only opposite directions give distance 2; if $r + s < 2$, no direction is feasible. The value π in that infeasible case is a convenient lower bound for the interval tests. At a zero radius, the definition requires no division and gives zero exactly when the other radius is at least 2.

For outer centers, the chord formula gives a required gap $2\eta(b)$, where

$$\eta(b) := \arcsin(1/b). \quad (2.7)$$

If $m \geq 2$, each consecutive outer gap is at least $2\eta(b)$, so their sum gives $m\eta(b) \leq \pi$. The same inequality holds when $m \leq 1$, since $0 < \eta(b) < \pi/6$. Thus in all cases

$$m\eta(b) \leq \pi. \quad (2.8)$$

For an inner radius, the required separation from any outer direction is

$$h_b(r) := g(r, b) = \begin{cases} 0, & 0 \leq r \leq b - 2, \\ \arccos \frac{r^2 + b^2 - 4}{2rb}, & b - 2 < r \leq b. \end{cases} \quad (2.9)$$

The larger domain up to b permits rational upper bounds on the inner radii to replace the exact cutoff.

We will use the monotonicity of h_b both to bound radii and to evaluate inequalities on intervals. The next statement gives the range where moving an inner center outwards increases its required angular separation.

Lemma 2.5. *For $b > 2$, the function h_b is nondecreasing on $[0, \sqrt{b^2 - 4}]$ and strictly increasing on $[b - 2, \sqrt{b^2 - 4}]$. Moreover, if*

$$r_*(b) := \sqrt{b^2 - 1} - \sqrt{3}, \quad (2.10)$$

then

$$b - 2 < r_*(b) < c(b), \quad h_b(r_*(b)) = \eta(b), \quad h_b(c(b)) = 2\eta(b). \quad (2.11)$$

Proof. For $b - 2 < r \leq b$, the cosine in (2.9) is

$$\chi_b(r) = \frac{1}{2b} \left(r + \frac{b^2 - 4}{r} \right). \quad (2.12)$$

For $b - 2 < r < s \leq \sqrt{b^2 - 4}$, subtraction gives

$$\chi_b(s) - \chi_b(r) = \frac{s - r}{2b} \left(1 - \frac{b^2 - 4}{rs} \right) < 0.$$

Arccosine decreases, and $h_b(b - 2) = 0$, so this proves the monotonicity assertion, including continuity at $b - 2$.

The cutoff satisfies $c(b)^2 = (b^2 - 4)(1 - 4/b^2) < b^2 - 4$. Substituting it into (2.12) gives $\chi_b(c(b)) = 1 - 2/b^2 = \cos(2\eta(b))$. Thus $h_b(c(b)) = 2\eta(b) > 0$. Continuity and strict monotonicity now give a unique radius between $b - 2$ and $c(b)$ where $h_b(r) = \eta(b)$. The equation for that radius is $r^2 - 2\sqrt{b^2 - 1}r + b^2 - 4 = 0$. Its two roots are $\sqrt{b^2 - 1} \pm \sqrt{3}$. Since $c(b) < \sqrt{b^2 - 1}$, the root in the required interval is the smaller one, namely $r_*(b)$. This proves all three identities. $\square$

2.5 Disjoint intervals of directions

The bound (2.8) uses only outer centers. We strengthen it by assigning each center an interval of directions and proving that the interiors of these intervals are disjoint. Their total length is then at most 2π , even though the intervals may cross the chosen zero direction.

Assign an outer center an interval with half-width $\eta(b)$, centered on its direction. For an inner center of radius r , use half-width

$$f_b(r) := \max\{h_b(r) - \eta(b), 0\}. \quad (2.13)$$

This choice makes an inner interval of positive length disjoint from every outer interval: the sum of their half-widths is exactly $h_b(r)$, the required separation. Outer intervals also have disjoint interiors, because the required outer–outer separation is $2\eta(b)$. An interval of zero length contributes nothing to the length sum.

Figure 3 shows the equality case for an inner and an outer interval. To complete the length bound, we must also check pairs of inner intervals; the following lemma reduces that check to four numerical inequalities.

Lemma 2.6 (Angular length bound). *Fix $b > 2$ and real bounds satisfying $0 < b_- < b$ and $0 \leq r_*^- < r_*(b) < c(b) < c_+$. Suppose that for all $r, s \in \{r_*^-, c_+\}$,*

$$r^2 + s^2 - 2rs(1 - 2/b_-^2) < 4. \quad (2.14)$$

If a projected feasible center set has m outer centers and inner radii $r_1, \dots, r_k$, then

$$m\eta(b) + \sum_{i=1}^k f_b(r_i) \leq \pi. \quad (2.15)$$

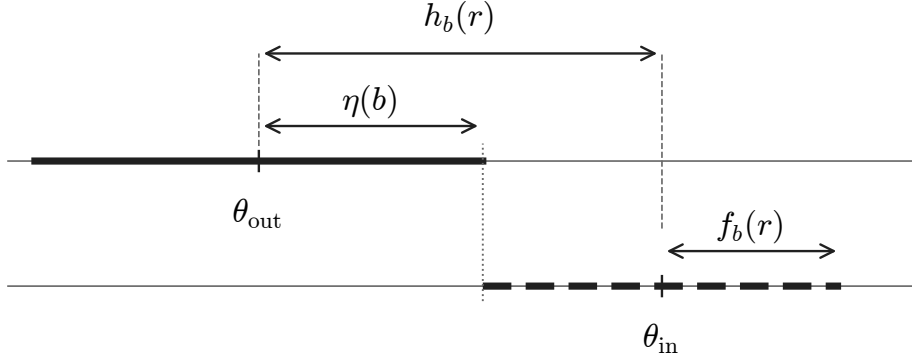


Figure 3: Intervals of directions assigned to an outer and an inner center, displayed on parallel angular axes. Their directions are θ_{out} and θ_{in} , and their half-widths are $\eta(b)$ and $f_b(r)$. When the directions differ by $h_b(r) = \eta(b) + f_b(r)$, the intervals meet at one endpoint.

Proof. By Lemma 2.5, a positive inner half-width requires $r_*(b) < r < c(b)$ and is less than $\eta(b)$. Consider two such radii. Since $\cos(2\eta(b)) = 1 - 2/b^2 > 1 - 2/b_-^2$, their squared distance at angular separation $2\eta(b)$ is at most $r^2 + s^2 - 2rs(1 - 2/b_-^2)$. Maximizing this quadratic in each radius in turn, as in the proof of Lemma 2.1, bounds it by its four corner values. They are less than 4 by (2.14), so the same holds throughout $[r_*^-, c_+]^2$.

The two inner directions must therefore differ by more than $2\eta(b)$, whereas their half-widths sum to less than $2\eta(b)$. Their intervals have disjoint interiors. We have already checked the outer–outer and outer–inner pairs. Summing all interval lengths gives $2m\eta(b) + 2\sum_i f_b(r_i) \leq 2\pi$, which is (2.15). $\square$

For a proposed inner count k , the remaining count is $m = N - k$. Lemma 2.6 therefore gives the necessary inequality

$$\sum_{i=1}^k f_b(r_i) \leq \pi - (N - k)\eta(b). \quad (2.16)$$

To exclude that count, it will suffice to prove that every separated set of k inner centers makes the left side strictly larger.

2.6 Preliminary bounds on the counts

Before making sharper exclusions, we need finite upper bounds for the number of inner centers. We obtain them by applying radial projection to the inner set alone, viewed as a separated set in a smaller disk. The following elementary estimate bounds the points that remain near the origin after that second projection.

Lemma 2.7 (A bound in a small disk). *Let $\mu \geq 2$ be an integer and let $0 \leq d < 2$. If*

$$\max\{d^2, 2d^2(1 - \cos(2\pi/\mu))\} < 4, \quad (2.17)$$

then a feasible center set in the disk of radius d has at most $\mu - 1$ centers.

Proof. If there were at least μ centers, choose μ of them. None could be O , because every other radius is less than 2. Order their directions around O , allowing ties. Some consecutive gap is at most $2\pi/\mu \leq \pi$, since the μ gaps sum to 2π . For that pair, monotonicity in the angle and Lemma 2.1, with both radial intervals $[0, d]$, bound the squared distance by the maximum in (2.17). This is less than 4, contrary to separation. Hence there are at most $\mu - 1$ centers. $\square$

Lemma 2.8 (Two radial layers). *Let $c_+ > 2$ and let $\nu \geq 2$ and $\mu \geq 2$ be integers. Put $d = c_+ - 4/c_+$ and suppose*

$$\nu \arcsin(1/c_+) > \pi, \quad 0 \leq d < 2,$$

with (2.17) holding for this d, μ . A feasible center set in radius c_+ then has at most $\nu + \mu - 2$ centers. Consequently, if $c(b) < c_+$, a projected feasible N -center set has $k \leq \nu + \mu - 2$ and $m \geq N - \nu - \mu + 2$.

Proof. Apply Lemma 2.2 with disk radius c_+ to the set being counted. There are at most $\nu - 1$ centers on its boundary: ν such centers would contradict the outer bound (2.8) with $b = c_+$. All other centers have radii less than d , so Lemma 2.7 bounds their number by $\mu - 1$. Adding gives $\nu + \mu - 2$.

For the last assertion, the original inner centers alone form a feasible set in radius c_+ . Apply the bound just proved to this subset and use $m = N - k$. The auxiliary projection serves only to bound the size of that subset. $\square$

We now have bounds on radial distances, angles and counts. The next restrictions use the gaps between consecutive outer directions to obtain more information about the inner centers.

3 Geometry between outer directions

Proof strategy. The outer centers divide the circle of directions into gaps. A large inner radius requires a large angle from each neighboring outer center, so an upper bound on a gap gives an upper bound on that radius. We first formalize this observation and the angle sum along a chain of centers. We then give a sharper result for five inner centers: under explicit numerical conditions, each outer gap contains at most one inner center's direction.

3.1 Gaps and chains of contacts

Fix a projected feasible center set in radius $b > 2$, with $m \geq 2$ outer centers. An *outer gap* is the closed interval of directions from one outer center to the next in counterclockwise order. Each gap has width at least $2\eta(b)$. To bound its possible excess, put

$$\lambda_m := \pi - m\eta(b), \quad \omega_m := \eta(b) + \lambda_m = \pi - (m - 1)\eta(b). \quad (3.1)$$

The outer count bound (2.8) gives $\lambda_m \geq 0$. If an outer gap has width δ , all the other $m - 1$ gaps therefore imply

$$\delta \leq 2\pi - 2(m - 1)\eta(b) = 2\omega_m. \quad (3.2)$$

Every direction in that gap is within ω_m of at least one endpoint. Hence an inner center of radius r_i satisfies

$$h_b(r_i) \leq \omega_m. \quad (3.3)$$

Here an assigned direction at O is harmless because $h_b(0) = 0$.

We also need the converse viewpoint: how much angle is required between two consecutive inner directions? Label the inner centers in cyclic order, and let q_i be the number of outer centers between inner center i and inner center $i + 1$. Indices on this cyclic list are read modulo k , so $\sum_i q_i = m$. Ties between inner directions may be ordered either way.

Lemma 3.1 (Angle along a chain). *Let r_i, r_{i+1} be consecutive inner radii and let δ_i be their counterclockwise angular gap. If $q_i \geq 1$, then*

$$\delta_i \geq \max\{g(r_i, r_{i+1}), h_b(r_i) + 2(q_i - 1)\eta(b) + h_b(r_{i+1})\}. \quad (3.4)$$

If $q_i = 0$, the bound $\delta_i \geq g(r_i, r_{i+1})$ still holds. In either case the sum of these lower bounds over i is at most 2π .

Proof. The smaller angular separation of the two inner centers is at least $g(r_i, r_{i+1})$, by (2.6). Their counterclockwise gap is no smaller. If $q_i \geq 1$, split that gap at each intervening outer direction. The first subgap is at least $h_b(r_i)$ and the last at least $h_b(r_{i+1})$, by (2.9). The $q_i - 1$ intermediate subgaps are each at least $2\eta(b)$. Adding proves the second bound in (3.4). Finally, the k counterclockwise inner gaps sum to one full turn, so the sum of their lower bounds is at most 2π . $\square$

Figure 4 illustrates the two kinds of subgap in the chain bound. Adding those subgaps gives exactly the second term of (3.4).

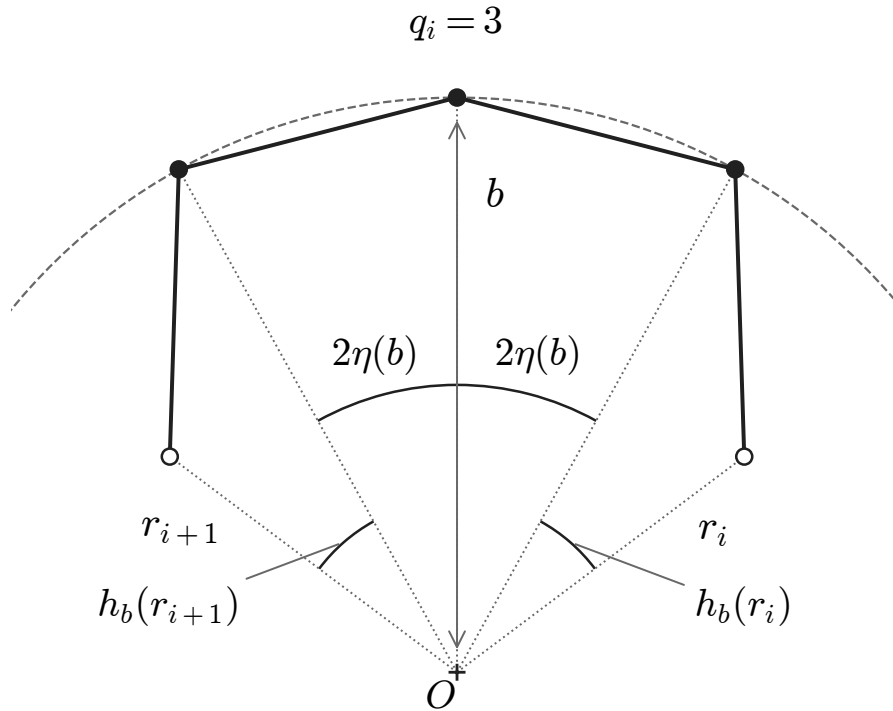


Figure 4: A contact chain with $q_i = 3$ outer centers (filled circles) between two inner centers (open circles). Solid segments have length 2; dotted rays show directions. The dashed arc has radius b . The first and last subgaps require $h_b(r_i)$ and $h_b(r_{i+1})$, and each intermediate subgap requires $2\eta(b)$.

If equality holds in the second bound of (3.4), every subgap equals its stated lower bound. If both inner radii exceed $b - 2$, these are positive contact angles. The cosine rule then makes every consecutive pair in the inner-outer-...-outer-inner chain a contact pair. This implication will recover exact positions after the inner radii have been determined.

3.2 An upper bound for an inner radius

An inner center must stay at distance at least 2 from its nearest outer neighbor. By (3.2), their angular separation is at most ω_m . In the range used below, increasing the inner radial distance decreases that center distance. Thus, if even the largest permitted angle gives distance less than 2 at a proposed upper bound r_{cap} , every larger inner radial distance is excluded. The lemma states inequalities that verify this argument using rational bounds for b and ω_m .

Lemma 3.2 (Radial bound from an outer gap). *Let a projected feasible center set in radius $b > 2$ have $m \geq 2$ outer centers. Let $b_- \leq b \leq b_+$ be positive bounds, and let $r_{\text{cap}} \geq 0$ and ω_m^+ satisfy*

$$0 < \omega_m \leq \omega_m^+ < \pi/2, \quad c(b) < b_- \cos \omega_m^+, \quad D(r_{\text{cap}}, b_+; \omega_m^+) < 4. \quad (3.5)$$

Then every inner radius is strictly less than r_{cap} .

Proof. Proof strategy. We bound the distance from an inner center to its nearest outer neighbor by increasing their angle, then decreasing the inner radial distance to r_{cap} , and finally increasing the outer radius to b_+ . Under the hypotheses, each change gives an upper bound for the squared distance, yet the final value is less than 4.

Fix an inner center and write r for its radial distance. Its nearest outer direction is at an angle at most ω_m , by (3.2). Since the two centers have distance at least 2 and D is nondecreasing in the angle on $[0, \pi]$,

$$4 \leq D(r, b; \omega_m) \leq D(r, b; \omega_m^+).$$

If $r_{\text{cap}} \geq c(b)$, the desired bound follows from $r < c(b)$. Suppose therefore that $r_{\text{cap}} < c(b)$, and assume for a contradiction that $r \geq r_{\text{cap}}$.

To compare values at the same outer radius b , let $t_{\text{cap}}, s_{\text{cap}}$ be any real numbers with $r_{\text{cap}} \leq t_{\text{cap}} < s_{\text{cap}} \leq c(b)$. Then

$$D(s_{\text{cap}}, b; \omega_m^+) - D(t_{\text{cap}}, b; \omega_m^+) = (s_{\text{cap}} - t_{\text{cap}})(t_{\text{cap}} + s_{\text{cap}} - 2b \cos \omega_m^+) < 0,$$

because $t_{\text{cap}} + s_{\text{cap}} \leq 2c(b) < 2b_- \cos \omega_m^+ \leq 2b \cos \omega_m^+$. Taking $t_{\text{cap}} = r_{\text{cap}}$ and $s_{\text{cap}} = r$ when they differ, and using equality when they agree, gives $D(r, b; \omega_m^+) \leq D(r_{\text{cap}}, b; \omega_m^+)$. The remaining comparison changes only the outer radius:

$$D(r_{\text{cap}}, b_+; \omega_m^+) - D(r_{\text{cap}}, b; \omega_m^+) = (b_+ - b)(b_+ + b - 2r_{\text{cap}} \cos \omega_m^+) \geq 0.$$

Here $b_+ \geq b$ and $r_{\text{cap}} < c(b) < b$, so both factors are nonnegative. The complete chain is consequently

$$4 \leq D(r, b; \omega_m) \leq D(r, b; \omega_m^+) \leq D(r_{\text{cap}}, b; \omega_m^+) \leq D(r_{\text{cap}}, b_+; \omega_m^+) < 4.$$

This contradiction proves $r < r_{\text{cap}}$ for every inner center. $\square$

3.3 Two-sided radius bounds for five inner centers

For exactly five inner centers, the angular length bound also gives a lower radius bound. A very small radius would force the other four radii to be large by (2.2); their four angular contributions would then exceed the available length. We define the two bounds so that this argument can be checked by explicit inequalities.

Place an outer center at $b\mathbf{e}(0)$, and consider a possible inner center $r\mathbf{e}(\theta)$ on the ray from O in direction θ . The angle θ is the separation between their directions. For $\theta \in [0, \pi/2)$ with $b \sin \theta \leq 2$, pair contact between these two centers requires

$$r^2 + b^2 - 2br \cos \theta = 4. \quad (3.6)$$

Its smaller radial root is

$$t_b(\theta) := b \cos \theta - \sqrt{4 - b^2 \sin^2 \theta}. \quad (3.7)$$

The radicand is nonnegative by the stated angle restriction, and the root is positive because $b^2 - 4 > 0$ and $b \cos \theta > 0$. It is the first intersection of the ray with the distance-2 circle about an outer center.

We choose the proposed upper bound so that its required angle from an outer direction is ω_m . For the proposed lower bound, the other four centers should each require an angular contribution $\lambda_m/4$ at the threshold. On the outer center's own ray, the near contact point is at radial distance $b - 2$; we also retain this value to describe the endpoints of the enclosure below. Using λ_m, ω_m from (3.1), define

$$\ell_{\text{gap}} := b - 2, \quad u_{\text{gap}} := t_b(\omega_m), \quad v_{\text{gap}} := 2 - t_b(\eta(b) + \lambda_m/4). \quad (3.8)$$

The bounds $v_{\text{gap}} \leq r_i \leq u_{\text{gap}}$ are claims to be proved in Lemma 3.3. The divisor 4 counts the four centers other than a chosen one; throughout this subsection and the next, the set has exactly five inner centers.

The following hypotheses give precisely the ranges needed for those claims. They concern the constants b, m , and are checked before applying the next two lemmas:

$$\begin{aligned} \lambda_m > 0, \quad 0 < v_{\text{gap}} < \ell_{\text{gap}} < r_*(b) < u_{\text{gap}} < c(b), \\ r_*(b) < 2 - v_{\text{gap}} < c(b), \quad u_{\text{gap}} < b \cos \omega_m, \\ b \sin \omega_m < 2, \quad 0 < \omega_m < \pi/2. \end{aligned} \quad (3.9)$$

Both angles in (3.8) then lie in $(0, \omega_m]$, so their roots exist. Since $u_{\text{gap}}, 2 - v_{\text{gap}} \in (r_*(b), c(b))$, the cosine rule in (3.6) and the definition of f_b give

$$\begin{aligned} h_b(u_{\text{gap}}) &= \omega_m, \\ h_b(2 - v_{\text{gap}}) &= \eta(b) + \lambda_m/4, \quad f_b(2 - v_{\text{gap}}) = \lambda_m/4. \end{aligned} \quad (3.10)$$

All angles here belong to $[0, \pi]$, so equality of cosines implies equality of the angles themselves.

Lemma 3.3 (Five inner radii). *Suppose a projected feasible set in radius $b > 2$ has five inner centers and $m \geq 2$ outer centers. Suppose it satisfies (2.15), and its constants satisfy (3.9). Then each inner radius satisfies*

$$v_{\text{gap}} \leq r_i \leq u_{\text{gap}} \quad (1 \leq i \leq 5). \quad (3.11)$$

Proof. By (3.3) and (3.10), $h_b(r_i) \leq \omega_m = h_b(u_{\text{gap}})$. If $r_i > u_{\text{gap}}$, strict monotonicity from Lemma 2.5 would reverse this inequality, because both radii lie in $(r_*(b), c(b))$. Thus $r_i \leq u_{\text{gap}}$.

For the lower bound, suppose $r_i < v_{\text{gap}}$. Each of the other four centers has radius $r_j \geq 2 - r_i > 2 - v_{\text{gap}}$, by (2.2). On $[2 - v_{\text{gap}}, c(b)]$, the function $f_b = h_b - \eta(b)$ is strictly increasing. Hence

$$\sum_{j \neq i} f_b(r_j) > 4f_b(2 - v_{\text{gap}}) = \lambda_m.$$

Adding the nonnegative term $f_b(r_i)$ contradicts (2.15), which bounds the whole sum by λ_m . Both bounds in (3.11) follow. $\square$

3.4 Five inner centers: at most one per outer gap

The radius bounds alone allow many positions inside an outer gap. To exclude two inner centers there, we enclose all permitted positions in the convex hull of five explicit vertices. The diameter of a convex hull is at most the largest distance between its vertices; we include that argument in the proof below.

Define the vertex set

$$\mathcal{V}_{\text{gap}} := \{\mathbf{u}_{\text{gap}}, \mathbf{l}_+, \mathbf{l}_-, \mathbf{v}_+, \mathbf{v}_-\}, \quad \begin{cases} \mathbf{u}_{\text{gap}} = (u_{\text{gap}}, 0), \\ \mathbf{l}_{\pm} = \ell_{\text{gap}} \mathbf{e}(\pm \omega_m), \\ \mathbf{v}_{\pm} = v_{\text{gap}} \mathbf{e}(\pm \omega_m). \end{cases} \quad (3.12)$$

Here the bold symbols denote vertices, whereas the corresponding italic symbols denote radial bounds. The *convex hull* is the set of all weighted averages of the vertices with nonnegative weights summing to 1.

Lemma 3.4 (One inner center per outer gap). *Under the hypotheses of Lemma 3.3, suppose that every distance between vertices of $\mathcal{V}_{\text{gap}}$ is strictly less than 2. Then each outer gap, including its endpoint directions, contains the direction of at most one inner center.*

Proof. Proof strategy. We first enlarge a chosen outer gap to width $2\omega_m$ and prove that every permitted inner position still satisfies the restrictions from its two endpoints. For the upper half of this enlarged gap, the distance and radius restrictions give four bounding lines. Their intersection lies in the convex hull of $\mathcal{V}_{\text{gap}}$. Reflection gives the same inclusion in the lower half, after which the vertex-distance hypothesis excludes two separated centers. Figure 5 shows the enlarged outer gap and the five vertices used in this proof.

Rotate the bisector of an outer gap to the positive horizontal axis, and write its endpoint angles as $-\alpha, \alpha$. By (3.2), $0 < \alpha \leq \omega_m$. Replace the two endpoints by

$$\mathbf{p}_- = b\mathbf{e}(-\omega_m), \quad \mathbf{p}_+ = b\mathbf{e}(\omega_m).$$

For a direction $\theta \in [-\alpha, \alpha]$, its angular distances from the endpoints increase from $\alpha \pm \theta$ to $\omega_m \pm \theta$. All these angles lie in $[0, \pi)$ because $2\omega_m < \pi$. By (2.1), each distance to an endpoint can only increase. Thus every permitted inner position is still at distance at least 2 from both enlarged endpoints.

Consider one such position (x, y) with $y \geq 0$. By Lemma 3.3, its radius is in $[v_{\text{gap}}, u_{\text{gap}}]$, and its direction is in $[0, \omega_m]$. The hypotheses (3.9) give

$$0 \leq y \leq u_{\text{gap}} \sin \omega_m < b \sin \omega_m < 2, \quad x \leq u_{\text{gap}} < b \cos \omega_m. \quad (3.13)$$

Since the point is to the left of $\mathbf{p}_+$ and its distance from that endpoint is at least 2, solving the distance inequality for x yields

$$x \leq F_{\text{gap}}(y) := b \cos \omega_m - \sqrt{4 - (y - b \sin \omega_m)^2}. \quad (3.14)$$

The radicand is positive because $0 < b \sin \omega_m - y < 2$ in (3.13). The direction restriction gives the other bound

$$x \geq y \cot \omega_m. \quad (3.15)$$

On the stated range of y , the function $F_{\text{gap}}(y)$ decreases, whereas $y \cot \omega_m$ increases. They agree at $y = \ell_{\text{gap}} \sin \omega_m$, since $b - \ell_{\text{gap}} = 2$ gives

$$F_{\text{gap}}(\ell_{\text{gap}} \sin \omega_m) = b \cos \omega_m - \sqrt{4 - 4 \sin^2 \omega_m} = \ell_{\text{gap}} \cos \omega_m.$$

Here $\cos \omega_m > 0$. Thus (3.14) and (3.15) can both hold only if

$$0 \leq y \leq \ell_{\text{gap}} \sin \omega_m. \quad (3.16)$$

We have bounded the height. We next replace the curved upper bound $x \leq F_{\text{gap}}(y)$ by a linear upper bound, then use the lower radial bound to complete the enclosure.

Both $\mathbf{u}_{\text{gap}}$ and $\mathbf{l}_+$ have distance 2 from $\mathbf{p}_+$. The first equality follows from the definition of t_b , and the second from $b - \ell_{\text{gap}} = 2$. Their segment lies in the closed disk of radius 2 about $\mathbf{p}_+$: the triangle inequality bounds the norm of every weighted average of the two radius vectors by 2. At each height in (3.16), the left boundary of this disk is $x = F_{\text{gap}}(y)$. Hence the segment's horizontal coordinate is at least $F_{\text{gap}}(y)$. Our permitted point lies on or to the left of that segment by (3.14).

The lower radius bound and $0 \leq \theta \leq \omega_m < \pi/2$ also give

$$x \geq v_{\text{gap}} \cos \omega_m.$$

Together with $y \geq 0$, the three line bounds now enclose the quadrilateral with vertices $(v_{\text{gap}} \cos \omega_m, 0)$, $\mathbf{u}_{\text{gap}}, \mathbf{l}_+, \mathbf{v}_+$. This quadrilateral is convex: it is the intersection of the four corresponding closed half-planes. Its extra vertex is the midpoint of $\mathbf{v}_+$ and $\mathbf{v}_-$. Consequently, every point of the quadrilateral is a convex combination of vertices in $\mathcal{V}_{\text{gap}}$. Reflection across the horizontal axis gives the same conclusion for permitted positions with $y \leq 0$.

Finally, write two permitted positions as $\mathbf{x} = \sum_i \alpha_i \mathbf{w}_i$ and $\mathbf{y} = \sum_j \beta_j \mathbf{w}_j$, where the $\mathbf{w}_i$ list the five vertices, the weights are nonnegative, and both weight sums are 1. The triangle inequality gives

$$\|\mathbf{x} - \mathbf{y}\| \leq \sum_{i,j} \alpha_i \beta_j \|\mathbf{w}_i - \mathbf{w}_j\| \leq \max_{i,j} \|\mathbf{w}_i - \mathbf{w}_j\| < 2. \quad (3.17)$$

Thus two separated inner centers cannot occur in the enlarged gap, and therefore cannot occur in the original gap. The argument includes coincident inner directions and directions on the endpoint rays. $\square$

Figure 5 depicts this enclosure. Its role is to show how the two distance-2 circles and the radial lower bound produce the five vertices; the inequalities above prove the inclusion.

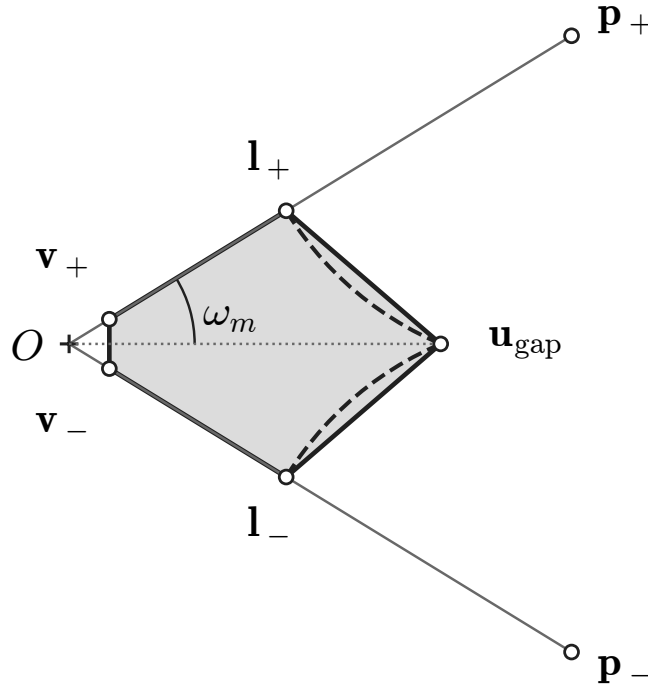


Figure 5: The enclosure for one outer gap after its bisector is horizontal and its width is enlarged to $2\omega_m$. The circular arcs have radius 2 about $\mathbf{p}_{\pm}$. All permitted inner positions lie in the convex hull of $\mathcal{V}_{\text{gap}}$.

The common gap lemmas now provide upper radius bounds, a two-sided bound when there are five inner centers, and a test that prevents two inner centers from sharing a gap. Along with the angular length bound, these will specify the starting cases of the finite computations.

4 Finite exclusion of radial intervals and cyclic orders

Proof strategy. We now specify how a finite calculation can exclude a whole family of feasible center sets. It records intervals for the inner radii and a cyclic order, while leaving the directions unknown. Necessary distance and angle inequalities either reject these data or restrict them further. Otherwise, subdivision replaces one radial interval by two smaller intervals whose union is the original. We prove that a completed calculation accounts for every real configuration in its starting ranges.

There are two uses of this procedure. To exclude an inner count k , we show that every separated k -point set violates (2.16). To restrict positions for a count still possible, we include the outer centers in the cyclic order and require every retained box to lie wholly within prescribed radial ranges. Throughout, every printed terminating decimal denotes an exact rational number; for example, $1.693 = 1693/1000$.

4.1 The data represented by a case

A *radial box* is a Cartesian product of closed intervals

$$\mathbf{I} = \prod_{i=1}^k I_i, \quad I_i = [r_i^-, r_i^+], \quad 0 \leq r_i^- \leq r_i^+. \quad (4.1)$$

It represents all simultaneous real choices of the k radii. For instance, $[1.6, 1.8] \times [1.7, 1.9]$ represents an entire rectangle of radius pairs. A valid exclusion must reject every point of that rectangle. Lemma 2.1 supplies the reason that some such exclusions can be certified at its corners.

The inner centers are labeled in cyclic order. Equal directions are allowed, and a center at O may be assigned any direction. When outer centers are included, the *cyclic count vector* $\mathbf{q}_{\text{out}} = (q_1, \dots, q_k)$ records the number of outer centers after inner center i and before inner center $i + 1$, as in Lemma 3.1. Its entries are nonnegative integers with sum m . Thus $(2, 1, 0)$ specifies three inner centers, with two outer centers in the first interval between them, one in the second, and none in the third. The vector specifies the order of the two types of center; the angular widths remain unknown. A *case* consists of such an order and a radial box.

4.2 Coverage by orders and symmetry reductions

Every projected feasible set admits a cyclic listing, including any chosen ordering of tied directions. Reading off the numbers of intervening outer centers gives a nonnegative vector with sum m . Conversely, listing all such vectors and allowing all radial choices covers every possible set. A geometric restriction may reduce this list, but it must have been proved for every feasible set before it is used.

Changing the starting inner center cyclically shifts the entries of $\mathbf{q}_{\text{out}}$. Reversing the direction of traversal reverses their cyclic order. More precisely, if the inner labels change from i to $k + 1 - i$, the new count after label i is q_{k-i} , with $q_0 = q_k$. The corresponding radii are relabeled by the same change of inner labels. Thus shifts and reversal generate the *dihedral equivalence* of the count vectors. We may retain one representative from each equivalence class, provided its radial intervals cover every relabeled radius vector. In the starting boxes used here, every radius initially has the same range. Rotations and reflections of the plane preserve all the distance conditions, so this reduction loses no feasible set.

For an inner-count exclusion, only the inner cyclic order is recorded. Choose a center of smallest radius as label 1. Reverse the traversal if needed to make the second radius no larger

than the last. Every set with $k \geq 2$ then has a labeling satisfying

$$r_1 \leq r_i \quad (1 \leq i \leq k), \quad r_2 \leq r_k. \quad (4.2)$$

Ties permit either choice. These inequalities can shorten intervals or exclude a box whose endpoints make them impossible. The count-vector calculations instead use their chosen dihedral representatives and retain every radial choice for those representatives.

4.3 Contracting a radial box

For any separated pair, (2.2) gives $r_i + r_j \geq 2$. Thus $r_i^+ + r_j^+ < 2$ excludes a box immediately. Otherwise, since $r_j \leq r_j^+$, every feasible radius satisfies $r_i \geq 2 - r_j^+$; the symmetric bound holds for r_j . The following interval updates therefore preserve every feasible point:

$$r_i^- \leftarrow \max\{r_i^-, 2 - r_j^+\}, \quad r_j^- \leftarrow \max\{r_j^-, 2 - r_i^+\}.$$

Applying these updates to all pairs can shorten several intervals. If a lower endpoint exceeds its upper endpoint, the box is empty and is excluded. The same rule applies to bounds obtained from the labeling inequalities (4.2).

4.4 Angle bounds valid throughout a box

For exact radii, (2.6) gives a required angle. For interval radii, we need one lower bound valid throughout a rectangle. The next lemma gives that bound from four certified endpoint bounds, using the squared-distance maximum already proved.

Lemma 4.1 (Four-corner angle bound). *Let $I = [r^-, r^+]$ and $J = [s^-, s^+]$ be closed intervals of nonnegative radii. For every corner (r', s') of $I \times J$, let $L(r', s') \in [0, \pi]$ be a lower bound for the smaller angular separation of any feasible pair at those radii. Put*

$$L := \min_{r' \in \{r^-, r^+\}, s' \in \{s^-, s^+\}} L(r', s').$$

Every pair of centers at distance at least 2, with radii in I, J , has smaller angular separation at least L . An arbitrary direction is permitted for a center at O .

Proof. Suppose instead that the smaller angle is $\theta < L$. At each corner, $\theta < L \leq L(r', s')$, so the assumed endpoint bound implies

$$D(r', s'; \theta) < 4.$$

By Lemma 2.1, the squared distance throughout the rectangle is no larger than the largest of these four values. It is therefore less than 4, contradicting separation. $\square$

The endpoint bounds are certified lower bounds for $g(r', s')$ in (2.6). Its degenerate conventions also satisfy the lemma: if $r' + s' < 2$, no direction gives distance 2, so the value π is valid; if $|r' - s'| \geq 2$, zero is valid for every direction. At $r' + s' = 2$ with positive radii, only angle π is feasible. These statements also cover zero radii without division.

On the nondegenerate branch, a proposed rational angle z is validated by the inequalities

$$0 \leq z \leq \pi, \quad \cos z \geq \frac{(r')^2 + (s')^2 - 4}{2r's'}.$$

Cosine decreases on $[0, \pi]$, so these imply $z \leq g(r', s')$. A rational polynomial lower bound for cosine makes the second inequality an exact rational test. Appendix B proves the polynomial bounds and their error estimates. If a numerically proposed angle fails certification, it supplies no premise; the calculation must use a certified smaller bound.

4.5 Directed angle inequalities and their closure

To combine bounds for a specified order of n centers, choose real representatives of the directions in one turn:

$$\theta_0 \leq \theta_1 \leq \cdots \leq \theta_{n-1} \leq \theta_0 + 2\pi.$$

The last inequality allows two centers on one ray to appear at opposite ends of the list. If $i < j$ and L_{ij} is a certified lower bound for the smaller angular separation, each of the two arcs between the centers has at least that length. Hence

$$\theta_j - \theta_i \geq L_{ij}, \quad \theta_i - \theta_j \geq L_{ij} - 2\pi. \quad (4.3)$$

For rational calculations, let Π^+ be a certified rational upper bound for 2π . Replacing the second right side by $L_{ij} - \Pi^+$ weakens the bound and preserves feasibility.

Write d_{ij} for a known lower bound on $\theta_j - \theta_i$. Adding the inequalities through any intermediate index a permits

$$d_{ij} \leftarrow \max\{d_{ij}, d_{ia} + d_{aj}\}. \quad (4.4)$$

Each update follows from two valid inequalities, so it remains valid throughout the radial box. A positive diagonal entry $d_{ii} > 0$ proves $0 = \theta_i - \theta_i \geq d_{ii} > 0$ and excludes the box. Initialize $d_{ii} = 0$, and, for every $i < j$, set $d_{ij} = L_{ij}$ and $d_{ji} = L_{ij} - \Pi^+$. Then let the intermediate index a run through $0, \dots, n-1$. For each a , let i run through $0, \dots, n-1$ and, for each i , let j run through the same range, applying (4.4) in place. After each intermediate index, reject the box if any diagonal entry is positive. These loops specify the directed closure used here, the longest-path form of the Floyd–Warshall algorithm. Every bound they return is valid by the addition argument above; that validity suffices for both rejection and the position bounds that follow.

For example, suppose the bounds for three directions in their listed order give $\theta_1 - \theta_0 \geq 1.2$ and $\theta_2 - \theta_1 \geq 1.1$. They imply $\theta_2 - \theta_0 \geq 2.3$. If other inequalities require the remaining arc back to the first direction to be at least 4.0, then $2\pi \geq 2.3 + 4.0 = 6.3$. This contradicts $2\pi < 6.284$, a consequence of the certified bound in Appendix B. The example illustrates how nonconsecutive pairs can combine to exclude an order that passes individual pair tests.

Even without a contradiction, the two directed bounds give useful position information:

$$d_{ij} \leq \theta_j - \theta_i \leq -d_{ji}. \quad (4.5)$$

Any two feasible sets with these radius ranges and this order obey the same interval for their relative directions. Once their reference directions are aligned, the angular difference of corresponding centers is at most the full width $-d_{ji} - d_{ij}$. This fact will convert radial localization into localization in the plane.

A shorter necessary test is available before this full closure. For each pair of consecutive inner centers, Lemma 3.1 gives a lower bound for their gap, including all intervening outer centers. Replace each term there by a certified lower bound over the radial intervals and add around the inner list. If the result exceeds an upper bound for 2π , the box is excluded immediately.

4.6 Testing an inner count

Fix a proposed inner count k and put $m = N - k$. Its necessary angular length bound is

$$\sum_{i=1}^k f_b(r_i) \leq \pi - m\eta(b), \quad (4.6)$$

by (2.16). Suppose the starting inner radii lie in $[0, c_+]$, where certified bounds satisfy

$$2 < b_- < b < b_+, \quad c(b) < c_+ < b_-, \quad c_+^2 < b_-^2 - 4. \quad (4.7)$$

The last inequality places $[0, c_+]$ in the monotonicity range of Lemma 2.5. Thus

$$\sum_i f_b(r_i) \geq \sum_i f_b(r_i^-). \quad (4.8)$$

We can test the necessary inequality using only lower bounds at these lower endpoints.

Let $\eta_- < \eta(b) < \eta_+$ be certified rational bounds, and let $\underline{g}$ denote a certified lower bound for g . For $0 \leq r \leq c_+$, we claim

$$f_b(r) \geq \max\{0, \underline{g}(r, b_+) - \eta_+\}. \quad (4.9)$$

For $r > 0$ on the nonzero branch of $g(r, b)$, the cosine quotient is $Q_r(b) = (b + (r^2 - 4)/b)/(2r)$. If $b_- \leq b_1 < b_2$, then

$$Q_r(b_2) - Q_r(b_1) = \frac{b_2 - b_1}{2r} \left(1 - \frac{r^2 - 4}{b_1 b_2} \right) > 0,$$

since $r \leq c_+ < b_-$. Arccosine therefore decreases as the outer radius increases. If the radial difference reaches 2, the required angle becomes zero and remains zero. The contribution at $r = 0$ is also zero. This proves $h_b(r) = g(r, b) \geq g(r, b_+)$ in every case, and hence (4.9).

The right side of (4.6) is at most $355/113 - m\eta_-$, using the certified upper bound $\pi < 355/113$. Combining this with (4.8) and (4.9) gives an exclusion whenever

$$\sum_i \max\{0, \underline{g}(r_i^-, b_+) - \eta_+\} > \frac{355}{113} - m\eta_-. \quad (4.10)$$

Radius contraction, labeling inequalities and the pairwise angle tests remain available as additional exclusions. A complete inner-count calculation must exclude every starting box compatible with (4.6).

4.7 Subdivision, acceptance and complete coverage

When the necessary inequalities do not exclude a case, choose one interval $[r_i^-, r_i^+]$ and a rational split point s_i strictly between its endpoints. Replace the case by two children using $[r_i^-, s_i]$ and $[s_i, r_i^+]$, with the other intervals unchanged. Their union is the original box. Both contain the split endpoint, so real radii at that endpoint remain covered.

A calculation can have either of two goals. An *exclusion* proves that no feasible set occurs in the starting cases; it must reject every final box. A *localization* proves that every feasible set satisfies one of a stated list of conditions, called *targets*. It can stop at a box once the whole box satisfies a target.

Each target specifies permitted cyclic orders and closed intervals for designated radial coordinates. It also lists the allowed correspondences between these coordinates and the center labels. Such a correspondence is an *allowed match*. A box is *accepted* if its order is permitted and, under one allowed match, every designated radial interval is contained in its target interval. Coordinates not designated by that target retain their current ranges. Containment of entire intervals ensures that every feasible set represented by the box satisfies the target.

Proposition 4.2 (Coverage of a completed calculation). *Suppose the starting orders and radial boxes cover every feasible center set under consideration. Suppose a finite completed calculation uses only the valid contractions, exclusions and subdivisions proved above. If every final box is either excluded or wholly contained in a stated target under an allowed match, then every feasible center set belongs to at least one target. If every final box is excluded, no such feasible set exists.*

Proof. Take a feasible set and choose a starting case containing its radii and order. Valid contractions preserve those radii, and a necessary inequality cannot exclude a box that still contains them. At each subdivision, at least one child contains the radius vector, including at a shared endpoint. Follow such a child at each step.

The completed calculation is finite, so this path reaches a final box. It cannot be an excluded box and therefore has an allowed match into a target. Its containment places the original set in that target as well. If there are no accepted final boxes, the existence of the original feasible set gives a contradiction. $\square$

The proposition is conditional on actual completion. A resource limit or an unresolved smallest interval gives no conclusion about the cases left over. Rational grids store interval endpoints; the intervals themselves retain all real radii between those endpoints. Exact inequalities and outward rounding certify every exclusion and containment test, with the arithmetic justified in Appendix B.

Each application below is a proposition with a proof headed Computation. That proof specifies the starting cases, tests, acceptance condition and completed output. Proposition 4.2 turns the output into a conclusion about every feasible center set in the starting cases. Appendix C records the case counts and terminal outcomes. Appendix H maps the propositions to the programs and data in the separate supplementary material, and states the trusted arithmetic base.

5 From nearby centers to exact agreement

Proof strategy. The interval calculations restrict radial distances and directions, but these ranges can still contain many positions. We first convert the ranges into bounds on the distance from each center to a specified reference center. We then give conditions under which a feasible set sufficiently close to the reference must agree with it after a rotation. Expanding the squared-distance constraints reduces this last step to two properties of linear equations, which we state and prove how to check.

5.1 Changes in a contact angle

The function $g(r, s)$ converts radii into a required angular separation. To control this angle when radii vary, we need its first and second derivatives. We work only where the three lengths $r, s, 2$ satisfy strict triangle inequalities, so the angle lies strictly between 0 and π .

Lemma 5.1 (Derivatives of the required angle). *For $r, s > 0$ with $|r - s| < 2 < r + s$, define*

$$\mathcal{D}(r, s) := 4r^2s^2 - (r^2 + s^2 - 4)^2. \quad (5.1)$$

Then $\mathcal{D}(r, s) > 0$, the function g is smooth, and

$$g_r(r, s) = -\frac{r^2 - s^2 + 4}{r\sqrt{\mathcal{D}(r, s)}}, \quad g_s(r, s) = -\frac{s^2 - r^2 + 4}{s\sqrt{\mathcal{D}(r, s)}}, \quad (5.2)$$

$$r^2 g_{rr}(r, s) = -\frac{16r^2(r^2 + s^2 - 4)}{\mathcal{D}(r, s)^{3/2}} + \frac{r^2 - s^2 + 4}{\sqrt{\mathcal{D}(r, s)}}, \quad (5.3)$$

$$rsg_{rs}(r, s) = \frac{32r^2s^2}{\mathcal{D}(r, s)^{3/2}}. \quad (5.4)$$

Exchanging r, s in (5.3) gives $s^2 g_{ss}$. Subscripts on g denote partial derivatives in the indicated arguments.

Proof. Proof strategy. We check the domain by factoring the discriminant, then differentiate the cosine identity once and twice. Positivity of the discriminant will justify every division.

The factorization

$$\mathcal{D}(r, s) = ((r + s)^2 - 4)(4 - (r - s)^2)$$

makes its positivity follow from the two strict triangle inequalities. The cosine-rule quotient is consequently in $(-1, 1)$, where inverse cosine is smooth. Also $\sin g(r, s) = \sqrt{\mathcal{D}(r, s)}/(2rs)$. Differentiating the defining cosine identity gives (5.2).

For the second derivatives, multiply the first identity by r . Differentiate $rg_r = -(r^2 - s^2 + 4)/\sqrt{\mathcal{D}}$ and use $\mathcal{D}_r = 4r(s^2 - r^2 + 4)$ to obtain

$$rg_r + r^2 g_{rr} = -\frac{16r^2(r^2 + s^2 - 4)}{\mathcal{D}^{3/2}}.$$

Subtracting rg_r gives (5.3). Differentiating rg_r in s instead, and using $\mathcal{D}_s = 4s(r^2 - s^2 + 4)$, gives (5.4). All denominators are positive on the stated domain. $\square$

We will use the mean value theorem in the form

$$|F(t_1) - F(t_0)| \leq M_{F,1}|t_1 - t_0| \quad (5.5)$$

when F is continuous on $[t_0, t_1]$, differentiable in its interior and $|F'| \leq M_{F,1}$. Here F denotes a real function of one variable, t_0, t_1 are its argument endpoints, and $M_{F,1}$ is a bound on the absolute derivative. A second derivative bound also controls the error after the linear term, as follows.

Lemma 5.2 (A Taylor remainder bound). *If F is twice continuously differentiable on $[0, 1]$ and $|F''| \leq M_{F,2}$, then*

$$|F(1) - F(0) - F'(0)| \leq M_{F,2}/2. \quad (5.6)$$

Proof. Put $e_F = F(1) - F(0) - F'(0)$ and $J_F(t) = F(t) - F(0) - tF'(0) - e_F t^2$. We have $J_F(0) = J_F(1) = J'_F(0) = 0$. Rolle's theorem gives a zero of J'_F in $(0, 1)$, and a second application gives a zero of J''_F between 0 and that point. At this second zero, $2e_F = F''$, which proves the bound. $\square$

5.2 Converting radial and angular errors to position errors

Let $\mathbf{x}_i^*$ be a specified reference center corresponding to a center $\mathbf{x}_i$. The star always distinguishes reference data from the unknown feasible set. To compare their positions, we will use a common bound b on both radii and bounds on their radial and angular differences.

Lemma 5.3 (Polar position estimate). *Let $\mathbf{x} = r\mathbf{e}(\theta)$ and $\mathbf{x}^* = r^*\mathbf{e}(\theta^*)$, with $0 \leq r, r^* \leq b$. If nonnegative constants $\varepsilon_r, \varepsilon_\theta$ satisfy $|r - r^*| \leq \varepsilon_r$ and $|\theta - \theta^*| \leq \varepsilon_\theta$, then*

$$\|\mathbf{x} - \mathbf{x}^*\| \leq \varepsilon_r + b\varepsilon_\theta. \quad (5.7)$$

The angle representatives may be any real numbers satisfying the stated difference bound.

Proof. Insert the intermediate point $r^*\mathbf{e}(\theta)$. The first segment has length $|r - r^*|$. The remaining chord has length at most the arc of angle $|\theta - \theta^*|$, because $2|\sin(t/2)| \leq |t|$. Hence

$$\begin{aligned} \|\mathbf{x} - \mathbf{x}^*\| &\leq |r - r^*| + r^*\|\mathbf{e}(\theta) - \mathbf{e}(\theta^*)\| \\ &\leq |r - r^*| + r^*|\theta - \theta^*| \leq \varepsilon_r + b\varepsilon_\theta. \end{aligned}$$

This also covers a zero radius, for which its assigned direction has no effect on the corresponding point. $\square$

For two feasible sets with a common order and radial box, (4.5) bounds the relative directions in both sets. Choose the same reference index a in each and rotate the unknown set so that the two reference directions agree. For center i , both relative angles belong to $[d_{ai}, -d_{ia}]$. We may therefore take

$$\varepsilon_{\theta,i} = -d_{ia} - d_{ai}, \quad \varepsilon_{r,i} = r_i^+ - r_i^-, \quad \varepsilon_{\text{pos},i} = \varepsilon_{r,i} + b\varepsilon_{\theta,i}. \quad (5.8)$$

The full interval widths bound the differences of any two entries in those intervals. For a corresponding pair of outer centers, both radii equal b , so its radial error is zero.

Figure 6 illustrates the two contributions in (5.7). We use the sum of their lengths to bound the Euclidean displacement and hence each Cartesian coordinate change.

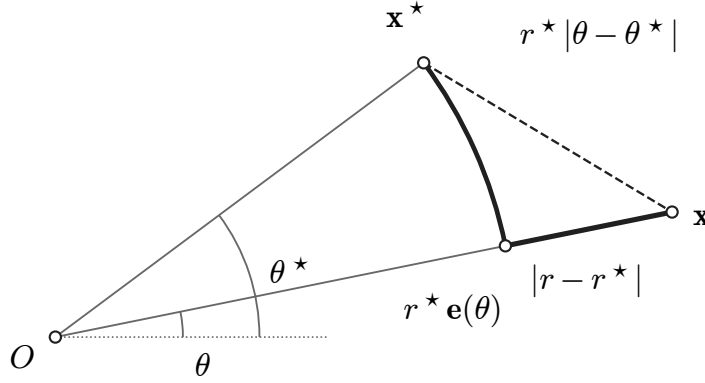


Figure 6: The polar position estimate. A radial segment of length $|r - r^*|$ and a circular arc of length $r^*|\theta - \theta^*|$ connect the two positions. Their combined length bounds the direct distance.

5.3 Linearized contact constraints

Fix a feasible reference set of n_{ref} labeled centers $\mathbf{x}_i^*$ in radius b , with $1 \leq i \leq n_{\text{ref}}$. We compare it with a feasible set whose center $\mathbf{x}_i$ has been assigned to reference center $\mathbf{x}_i^*$ for each i . The assignment of labels is part of this comparison. Select some reference pair contacts, for which $\|\mathbf{x}_i^* - \mathbf{x}_j^*\| = 2$, and some reference container contacts, for which $\|\mathbf{x}_i^*\| = b$. In the compared set, the same pairs must have distance at least 2 and the same centers must have norm at most b .

A rotated copy of the reference is also feasible. To distinguish actual movement from this common rotation, fix an index i_0 and impose one coordinate condition on the displacements:

$$\mathbf{v}_i := \mathbf{x}_i - \mathbf{x}_i^* = (v_{i,x}, v_{i,y}), \quad U := \max_i \{|v_{i,x}|, |v_{i,y}|\}, \quad v_{i_0,x} = 0. \quad (5.9)$$

Here $\mathbf{v}_i$ is a plane vector and U is a nonnegative real number. Exact agreement of all corresponding centers means $U = 0$. The criterion below assumes the displayed coordinate condition.

In applications, the selected corresponding centers $\mathbf{x}_{i_0}$ and $\mathbf{x}_{i_0}^*$ both have norm b . A rotation aligns their directions and hence makes their positions equal: both $v_{i_0,x}$ and $v_{i_0,y}$ are zero. We use only the first equality in the criterion, allowing the second coordinate to remain among the variables of the linear equations. This proves a criterion for a larger class of displacements, which includes the aligned set. Choose the axes so that the vertical coordinate of $\mathbf{x}_{i_0}^*$ is nonzero. A rotation by a small angle has first-order horizontal displacement equal to minus that coordinate times the angle, so fixing the horizontal coordinate removes the first-order rotational motion.

For a selected pair contact i, j , expand the squared distance:

$$\|\mathbf{x}_i - \mathbf{x}_j\|^2 = 4 + 2(\mathbf{x}_i^* - \mathbf{x}_j^*) \cdot (\mathbf{v}_i - \mathbf{v}_j) + \|\mathbf{v}_i - \mathbf{v}_j\|^2.$$

The dot denotes the Euclidean scalar product. The middle term is linear in the displacements; define this *linearized constraint* by

$$y_{ij} := 2(\mathbf{x}_i^* - \mathbf{x}_j^*) \cdot (\mathbf{v}_i - \mathbf{v}_j). \quad (5.10)$$

Feasibility requires the squared distance to be at least 4. Each coordinate of $\mathbf{v}_i - \mathbf{v}_j$ has absolute value at most $2U$, so

$$y_{ij} \geq -\|\mathbf{v}_i - \mathbf{v}_j\|^2 \geq -8U^2. \quad (5.11)$$

For a selected container contact i , expansion of containment gives

$$b^2 \geq \|\mathbf{x}_i^* + \mathbf{v}_i\|^2 = b^2 + 2\mathbf{x}_i^* \cdot \mathbf{v}_i + \|\mathbf{v}_i\|^2.$$

Moving the linear term to the left gives a lower bound, as for the pair constraints. We therefore define the container expression with a minus sign, $y_i^{\text{bd}} := -2\mathbf{x}_i^* \cdot \mathbf{v}_i$, and obtain

$$y_i^{\text{bd}} \geq \|\mathbf{v}_i\|^2 \geq 0. \quad (5.12)$$

The subscript pair ij in y_{ij} records two touching disks; the single subscript in y_i^{bd} records a disk touching the container boundary.

To handle both kinds together, enumerate all selected expressions by $a = 1, \dots, n_{\text{con}}$ and write their values as the column $\mathbf{y}_{\text{lin}} = (y_a)$. Each y_a is one selected y_{ij} or one selected y_i^{bd} . Equations (5.11) and (5.12) give the common bound

$$y_a \geq -8U^2 \quad (1 \leq a \leq n_{\text{con}}). \quad (5.13)$$

Thus feasibility bounds every linear expression from below by a quadratic error. We next give conditions that also bound them from above and force all displacements to vanish.

5.4 A positive equilibrium stress and a quantitative bound

The lower bounds (5.13) alone permit large positive values of y_a . To obtain upper bounds, suppose that a fixed positive weighted sum of these expressions is always zero. If one expression is positive, the others must cancel its weighted value. Since each of the others is at least $-8U^2$, that cancellation bounds the positive value by a constant times U^2 .

We also need a bound on the displacements in terms of the linear expressions. If such a bound holds, the preceding argument gives $U \leq C_{\text{loc}}U^2$ for a positive constant C_{loc} . A positive U satisfying this inequality is at least $1/C_{\text{loc}}$, so every sufficiently small feasible displacement must be zero. The following lemma assumes both required properties and gives the resulting threshold explicitly. A fixed list of weights whose linear combination vanishes for every allowed displacement is called an *equilibrium stress*.

Lemma 5.4 (Quantitative local rigidity). *For a reference set and selected constraints as above, suppose $n_{\text{con}} \geq 2$. Let $\kappa > 0$ and $\sigma_* > 0$ be constants. The first hypothesis is a positive equilibrium stress: fixed weights w_a satisfy*

$$w_a > \sigma_*, \quad \sum_{a=1}^{n_{\text{con}}} w_a = 1, \quad \sum_{a=1}^{n_{\text{con}}} w_a y_a = 0 \quad (5.14)$$

for every coordinate change with $v_{i_0, x} = 0$. The second hypothesis is a bound for the coordinate changes: all such changes satisfy

$$U \leq \kappa \max_a |y_a|. \quad (5.15)$$

Then any corresponding feasible set with $v_{i_0,x} = 0$ and

$$U < \frac{\sigma_*}{8\kappa} \quad (5.16)$$

coincides with the reference set at every labeled center.

Proof. Proof strategy. We first use the positive weighted identity to turn the lower bounds $y_a \geq -8U^2$ into upper bounds. The coordinate bound (5.15) will then bound U by a constant times U^2 . The stated threshold excludes a positive solution of that inequality.

Fix an index a . Isolate its term in (5.14) and use (5.13) for the others:

$$w_a y_a = - \sum_{d \neq a} w_d y_d \leq 8U^2 \sum_{d \neq a} w_d = 8(1 - w_a)U^2.$$

Division by $w_a > \sigma_*$ gives $y_a \leq 8U^2/\sigma_*$. Since positive weights sum to 1 and there are at least two, $0 < \sigma_* < 1$. The lower bound $y_a \geq -8U^2$ therefore gives

$$\max_a |y_a| \leq \frac{8U^2}{\sigma_*}.$$

Substituting into (5.15) yields

$$U \leq \frac{8\kappa}{\sigma_*} U^2.$$

If $U > 0$, division gives $U \geq \sigma_*/(8\kappa)$, contrary to (5.16). Thus $U = 0$, and every displacement in (5.9) vanishes. $\square$

The conclusion is called *local rigidity*: after the alignment, all feasible sets within a specified neighborhood agree with the reference. The stress identity (5.14) and the coordinate bound (5.15) also imply *infinitesimal rigidity*: the only allowed displacement satisfying all linearized feasibility inequalities $y_a \geq 0$ is zero. Indeed, positivity of every weight in (5.14) forces every nonnegative y_a to vanish; then (5.15) gives $U = 0$. The relation between infinitesimal and local rigidity is established in the general framework literature [14, Section 2.2]; Lemma 5.4 supplies the explicit bound needed here and proves it directly.

5.5 Finding a stress and a coordinate bound by matrix inversion

Lemma 5.4 requires the positive weighted identity (5.14) and the coordinate bound (5.15). We now show how one matrix inverse can supply both. The construction of the matrix uses the known reference coordinates; its invertibility, its inverse bound and the positivity of the weights must then be proved for that reference.

Writing the linear expressions as a matrix. To find weights that cancel every linear expression, we first record the coefficient of every displacement coordinate. List $v_{1,x}, v_{1,y}, \dots, v_{n_{\text{ref}},x}, v_{n_{\text{ref}},y}$ in that order and delete the fixed zero coordinate $v_{i_0,x}$. The remaining $n_{\text{var}} = 2n_{\text{ref}} - 1$ coordinates form a column $\mathbf{v}'$. Let $\mathbf{A}$ have one row for each selected expression y_a and one column for each entry of $\mathbf{v}'$; its entries are the corresponding coefficients. Thus

$$\mathbf{y}_{\text{lin}} = \mathbf{A}\mathbf{v}'. \quad (5.17)$$

This coefficient matrix is the *signed rigidity matrix* of the selected constraints; the signs agree with the lower bounds in (5.13).

Write the reference coordinates as $\mathbf{x}_i^* = (x_{i,x}^*, x_{i,y}^*)$. For a pair contact i, j , expanding (5.10) gives

$$y_{ij} = 2(x_{i,x}^* - x_{j,x}^*)v_{i,x} + 2(x_{i,y}^* - x_{j,y}^*)v_{i,y} \\ - 2(x_{i,x}^* - x_{j,x}^*)v_{j,x} - 2(x_{i,y}^* - x_{j,y}^*)v_{j,y}.$$

These four coefficients occupy the columns for the two centers; all other coefficients are zero. For a container contact i , the expression $y_i^{\text{bd}} = -2x_{i,x}^*v_{i,x} - 2x_{i,y}^*v_{i,y}$ gives two coefficients, with zeros in every other column. Deleting the column for $v_{i_0,x} = 0$ does not change either expression.

The equations for the weights. Let $\mathbf{w}_{\text{stress}} = (w_a)$ be a column of proposed weights. Their weighted sum is $\mathbf{w}_{\text{stress}}^T \mathbf{y}_{\text{lin}} = \mathbf{w}_{\text{stress}}^T \mathbf{A} \mathbf{v}'$. It vanishes for every $\mathbf{v}'$ exactly when every coefficient in the row $\mathbf{w}_{\text{stress}}^T \mathbf{A}$ is zero. Normalizing the weights therefore asks for the two equations

$$\mathbf{w}_{\text{stress}}^T \mathbf{A} = 0, \quad \mathbf{w}_{\text{stress}}^T \mathbf{1} = 1,$$

where $\mathbf{1}$ is the column of n_{con} ones. These equations explain the column of ones in the following square matrix. Assume $n_{\text{con}} \geq n_{\text{var}} + 1$; if more columns are needed, choose a rational matrix $\mathbf{V}$ with $n_{\text{con}} - n_{\text{var}} - 1$ columns and put

$$\mathbf{H} = [\mathbf{A} \mid \mathbf{1} \mid \mathbf{V}]. \quad (5.18)$$

When $n_{\text{con}} = n_{\text{var}} + 1$, the last block is empty. We seek a choice of $\mathbf{V}$ for which $\mathbf{H}$ is invertible; this property is an additional condition to check.

The new columns also allow us to recover displacements from the linear expressions. For an actual displacement, append $n_{\text{con}} - n_{\text{var}}$ zero entries to $\mathbf{v}'$. Then

$$\mathbf{H} \begin{pmatrix} \mathbf{v}' \\ 0 \\ 0 \end{pmatrix} = \mathbf{A} \mathbf{v}' + \mathbf{1} 0 + \mathbf{V} 0 = \mathbf{y}_{\text{lin}}.$$

The single zero multiplies the column $\mathbf{1}$, and the last zero column has the number of entries required by $\mathbf{V}$. If $\mathbf{H}$ is invertible, its inverse recovers this padded column from $\mathbf{y}_{\text{lin}}$. The next proposition proves both uses of the inverse and states the bounds they require.

Proposition 5.5. *Suppose $\mathbf{H}$ is invertible. If every absolute row sum of $\mathbf{H}^{-1}$ is at most κ , then (5.15) holds. Row $n_{\text{var}} + 1$ of $\mathbf{H}^{-1}$ gives weights satisfying $\sum_a w_a = 1$ and $\sum_a w_a y_a = 0$ for all allowed coordinate changes. If each entry of this row exceeds σ_* , the weights satisfy (5.14).*

Proof. Proof strategy. We apply the inverse to the padded displacement column to obtain the coordinate bound. We then multiply its row corresponding to the column of ones by $\mathbf{H}$ to obtain the normalized weights.

For the vector in (5.17), a known solution of $\mathbf{H} \mathbf{z}_{\text{lin}} = \mathbf{y}_{\text{lin}}$ is

$$\mathbf{H}^{-1} \mathbf{y}_{\text{lin}} = \begin{pmatrix} \mathbf{v}' \\ 0 \end{pmatrix}.$$

Indeed, multiplying the displayed column by $\mathbf{H}$ gives $\mathbf{A} \mathbf{v}' = \mathbf{y}_{\text{lin}}$, and the solution is unique. The absolute row-sum bound therefore gives

$$U = \max_i |v'_i| \leq \kappa \max_a |y_a|,$$

since the omitted coordinate is zero.

Let $\mathbf{w}_{\text{stress}}^T$ be row $n_{\text{var}} + 1$ of the inverse. Multiplying it by the first $n_{\text{var}} + 1$ columns of $\mathbf{H}$ gives, from $\mathbf{H}^{-1} \mathbf{H} = \text{Id}$,

$$\mathbf{w}_{\text{stress}}^T \mathbf{A} = 0, \quad \mathbf{w}_{\text{stress}}^T \mathbf{1} = 1.$$

The first identity cancels the linearized constraints for every $\mathbf{v}'$, and the second normalizes the weights. The additional entry bound is precisely their required positivity. $\square$

Example: why the column of ones gives weights. Consider two scalar variables z_1, z_2 and the three expressions $y_1 = z_1$, $y_2 = z_2$, $y_3 = -z_1 - z_2$. This example isolates the linear algebra in Proposition 5.5. The three expressions always sum to zero, so weights $1/3, 1/3, 1/3$ cancel them and sum to 1. We now obtain those weights from an inverse.

The coefficient matrix and its augmentation are

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{pmatrix}.$$

For this example only, these symbols denote the displayed matrices. Introduce one auxiliary scalar s_{aux} , which multiplies the new column. For any prescribed triple (y_1, y_2, y_3) , solve

$$z_1 + s_{\text{aux}} = y_1, \quad z_2 + s_{\text{aux}} = y_2, \quad -z_1 - z_2 + s_{\text{aux}} = y_3. \quad (5.19)$$

Adding the equations gives the first formula below; substituting it into the first two equations gives the others:

$$s_{\text{aux}} = \frac{y_1 + y_2 + y_3}{3}, \quad z_1 = \frac{2y_1 - y_2 - y_3}{3}, \quad z_2 = \frac{-y_1 + 2y_2 - y_3}{3}.$$

These formulas are the rows of H^{-1} applied to the prescribed triple. For a triple arising from the original expressions, $y_1 + y_2 + y_3 = 0$, so $s_{\text{aux}} = 0$. The row that computes it is exactly the normalized stress $(1/3, 1/3, 1/3)$. The other two rows have absolute sums $4/3$; hence the triangle inequality gives

$$\max\{|z_1|, |z_2|\} \leq \frac{4}{3} \max\{|y_1|, |y_2|, |y_3|\}.$$

Thus the augmented inverse gives both the cancelling weights and a bound for the two variables, as required by the general method.

5.6 Proving an inverse bound from its residual

The reference coordinates may be irrational. We therefore choose a rational matrix P and prove that it approximates an actual inverse, using bounds on the *residual*

$$E := \text{Id} - PH. \quad (5.20)$$

The proof below establishes invertibility as well as the error bound. An approximate inverse is consequently a proposed certificate whose validity is checked by exact inequalities.

In the following estimates we consider a linear system $H\mathbf{z}_{\text{lin}} = \mathbf{y}_{\text{lin}}$. The real column $\mathbf{y}_{\text{lin}}$ is prescribed, and the real column $\mathbf{z}_{\text{lin}}$ is the unknown solution. For a real vector, $\|\cdot\|_{\infty}$ denotes its largest absolute entry; for a real matrix, it denotes the largest absolute row sum. The latter is the induced norm: rowwise triangle inequalities give

$$\|M\mathbf{z}_{\text{lin}}\|_{\infty} \leq \|M\|_{\infty} \|\mathbf{z}_{\text{lin}}\|_{\infty}, \quad \|MN\|_{\infty} \leq \|M\|_{\infty} \|N\|_{\infty}. \quad (5.21)$$

For the second inequality, row i has absolute sum at most $\sum_d |M_{id}| \sum_j |N_{dj}|$, which is at most $\sum_d |M_{id}| \|N\|_{\infty}$.

Lemma 5.6 (Residual certificate). *Let H, P be real square matrices of the same size. Suppose $0 \leq \rho_{\text{inv}} < 1$ and*

$$\|\text{Id} - PH\|_{\infty} \leq \rho_{\text{inv}}. \quad (5.22)$$

Then $\mathbf{H}$ is invertible and

$$\|\mathbf{H}^{-1}\|_{\infty} \leq \frac{\|\mathbf{P}\|_{\infty}}{1 - \rho_{\text{inv}}}, \quad (5.23)$$

while

$$\|\mathbf{H}^{-1} - \mathbf{P}\|_{\infty} \leq \varepsilon_{\text{inv}} := \frac{\rho_{\text{inv}} \|\mathbf{P}\|_{\infty}}{1 - \rho_{\text{inv}}}. \quad (5.24)$$

Proof. Proof strategy. We first show that the homogeneous system has only the zero solution, which proves invertibility. We then bound solutions for any prescribed column $\mathbf{y}_{\text{lin}}$, and use that bound to estimate the inverse error. All divisions below are permitted by $1 - \rho_{\text{inv}} > 0$.

If $\mathbf{H}\mathbf{z}_{\text{lin}} = 0$, the residual identity gives $\mathbf{z}_{\text{lin}} = \mathbf{E}\mathbf{z}_{\text{lin}}$. Equations (5.21) and (5.22) imply $\|\mathbf{z}_{\text{lin}}\|_{\infty} \leq \rho_{\text{inv}} \|\mathbf{z}_{\text{lin}}\|_{\infty}$, so $\mathbf{z}_{\text{lin}} = 0$. A square matrix with trivial kernel is invertible, by Gaussian elimination.

For $\mathbf{H}\mathbf{z}_{\text{lin}} = \mathbf{y}_{\text{lin}}$, the same identity gives $\mathbf{z}_{\text{lin}} = \mathbf{P}\mathbf{y}_{\text{lin}} + \mathbf{E}\mathbf{z}_{\text{lin}}$. Moving the residual term in the norm estimate to the left yields

$$(1 - \rho_{\text{inv}}) \|\mathbf{z}_{\text{lin}}\|_{\infty} \leq \|\mathbf{P}\|_{\infty} \|\mathbf{y}_{\text{lin}}\|_{\infty}. \quad (5.25)$$

To bound one absolute row sum of $\mathbf{H}^{-1}$, choose each y_j to be the sign of the corresponding entry in that row, using zero for a zero entry. Then $\|\mathbf{y}_{\text{lin}}\|_{\infty} \leq 1$, and the corresponding entry of $\mathbf{z}_{\text{lin}} = \mathbf{H}^{-1}\mathbf{y}_{\text{lin}}$ equals that row's absolute sum. Equation (5.25) proves (5.23) for every row.

Finally, multiplication of the residual identity by $\mathbf{H}^{-1}$ gives $\mathbf{E}\mathbf{H}^{-1} = \mathbf{H}^{-1} - \mathbf{P}$. The product bound (5.21) and (5.23) give (5.24). This proves invertibility and both asserted bounds. $\square$

Each entry error is bounded by ε_{inv} , since an entry is no larger in absolute value than its row sum. Consequently, for the augmented matrix (5.18), it suffices to certify the rational inequalities

$$\frac{\|\mathbf{P}\|_{\infty}}{1 - \rho_{\text{inv}}} < \kappa, \quad \min_a P_{n_{\text{var}}+1,a} - \varepsilon_{\text{inv}} > \sigma_*. \quad (5.26)$$

Proposition 5.5 then proves the positive stress identity (5.14) and the coordinate bound (5.15) required by Lemma 5.4. The entries of $\mathbf{H}$ can be enclosed in rational intervals, so multiplication by the rational matrix $\mathbf{P}$ bounds each residual entry by exact arithmetic.

The local method is now complete. A common radial box and order give position bounds through (5.8); a residual certificate gives κ and σ_* . If every coordinate error is below $\sigma_*/(8\kappa)$, the nearby feasible set agrees exactly with its aligned reference.

6 Constructing centers from prescribed contacts

We will specify exact packings by successive contacts. Two elementary coordinate operations make those specifications explicit: rotation between two centers on radius b at mutual distance 2, and intersection of two circles of radius 2. We record their formulas here for use in any construction.

Let $\mathbf{J}(x, y) = (-y, x)$ be the counterclockwise quarter-turn, and let $\mathbf{S}(x, y) = (-x, y)$ be reflection in the vertical axis. To step between boundary points at distance 2, put

$$\tau_b := \sqrt{b^2 - 1}, \quad \mathbf{R}_b = \begin{pmatrix} 1 - 2/b^2 & -2\tau_b/b^2 \\ 2\tau_b/b^2 & 1 - 2/b^2 \end{pmatrix}. \quad (6.1)$$

The sum of the squares of either column is 1, the columns are orthogonal, and the determinant is 1. Its entries are $\cos(2\eta(b))$ and $\sin(2\eta(b))$, so it rotates through $2\eta(b)$. In particular, if $\|\mathbf{x}\| = b$, then $\|\mathbf{R}_b\mathbf{x}\| = b$ and $\|\mathbf{R}_b\mathbf{x} - \mathbf{x}\|^2 = 2b^2(2/b^2) = 4$.

For distinct points $\mathbf{x}, \mathbf{y}$ with $\|\mathbf{y} - \mathbf{x}\| \leq 4$, define

$$\mathbf{l}_{\pm}(\mathbf{x}, \mathbf{y}) := \frac{\mathbf{x} + \mathbf{y}}{2} \pm \sqrt{\frac{4}{\|\mathbf{y} - \mathbf{x}\|^2} - \frac{1}{4}} \mathbf{J}(\mathbf{y} - \mathbf{x}). \quad (6.2)$$

These are precisely the points at distance 2 from both arguments. Indeed, subtracting the two squared-distance equations puts the point on their perpendicular bisector. Its displacement from the midpoint is perpendicular to $\mathbf{y} - \mathbf{x}$, and Pythagoras requires its length to be $\sqrt{4 - \|\mathbf{y} - \mathbf{x}\|^2/4}$, giving the formula. The plus sign chooses the left side of the directed line from $\mathbf{x}$ to $\mathbf{y}$; the minus sign chooses the right. The two points coincide when the arguments are distance 4 apart.

These operations determine centers from prescribed pair contacts and container contacts. Their formulas also express every required square root and denominator explicitly.

7 Fifteen circles

Proof strategy. We first construct a feasible set using a pentagon and five squares. For an arbitrary feasible set at that radius, the shared geometric bounds and two finite computations will leave five inner centers, ten outer centers, and narrow ranges for the inner radii. We then compare two lower bounds for each angular gap. Their sum forces all five radii to equal the pentagon radius, after which equality fixes the directions and recovers every original center.

7.1 A pentagon with five outward squares

The construction begins with a regular pentagon of side 2, centered at O . Erect a square outward on each side. The five pentagon vertices and the ten additional square vertices will give fifteen centers. To calculate their radii, put

$$\varphi := \pi/5, \quad a := \csc \varphi, \quad t := \cot \varphi. \quad (7.1)$$

A right triangle from O to a side midpoint and an adjacent vertex has opposite side 1 and angle φ at O . Thus the pentagon circumradius is a and its apothem is t .

Place one side midpoint on the positive horizontal axis. The square on that side has inner and outer vertices

$$\mathbf{A}_{\pm} = (t, \pm 1), \quad \mathbf{P}_{\pm} = (t + 2, \pm 1). \quad (7.2)$$

The outer vertices have common radius

$$b_{15} := \sqrt{(t + 2)^2 + 1}. \quad (7.3)$$

Rotating this square by multiples of 2φ defines the whole center set explicitly as

$$\begin{aligned} \mathcal{P}_{15} := & \{a\mathbf{e}((2j + 1)\varphi) : 0 \leq j \leq 4\} \\ & \cup \{(t + 2)\mathbf{e}(2j\varphi) \pm \mathbf{J}\mathbf{e}(2j\varphi) : 0 \leq j \leq 4\}. \end{aligned}$$

Figure 7 shows the squares and the right triangle used to determine these radii.

To express the radius in radicals, put $\gamma = \cos \varphi$. Since $3\varphi = \pi - 2\varphi$, the triple- and double-angle identities give $(\gamma + 1)(4\gamma^2 - 2\gamma - 1) = 0$. The unique positive root is $(1 + \sqrt{5})/4$. Substituting into $\csc^2 \varphi$ and $\cot^2 \varphi$ gives

$$a^2 = 2 + 2/\sqrt{5}, \quad t^2 = 1 + 2/\sqrt{5}, \quad (7.4)$$

and hence

$$b_{15}^2 = 6 + 2/\sqrt{5} + 4\sqrt{1 + 2/\sqrt{5}}. \quad (7.5)$$

These identities also give $a < 2$ and $1 < t < 3/2$. The first will compare boundary gaps; both will be used in the lower-bound argument.

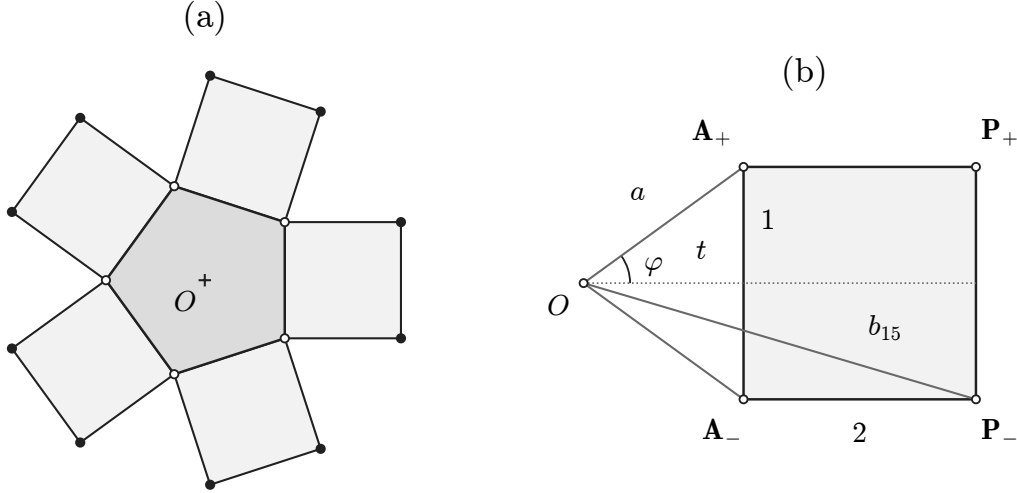


Figure 7: The fifteen-center construction. The five vertices of the regular pentagon have radius a . Each outward square contributes two vertices of radius b_{15} . For the illustrated side, its midpoint has radius t and each half-side has length 1.

Lemma 7.1. *The set $\mathcal{P}_{15}$ is feasible in radius b_{15} . Its contact angles satisfy*

$$g(a, a) = 2\varphi, \quad h_{b_{15}}(a) = \varphi - \eta(b_{15}). \quad (7.6)$$

Proof. Proof strategy. We check inner pairs, outer pairs and mixed pairs separately. The square coordinates give the contacts; the angular ordering will show that every remaining pair is at least as far apart. All fifteen centers have radius at most b_{15} by (7.3) and $a^2 = t^2 + 1$.

The five inner centers form a regular pentagon of side 2. Its diagonals have larger smaller angles than its sides, so they are longer by (2.1). For the outer centers, (7.2) shows that their directions are $2j\varphi \pm \eta(b_{15})$. Their successive gaps alternate between $2\eta(b_{15})$ and $2\varphi - 2\eta(b_{15})$. The half-angle identity gives

$$\cot(\varphi/2) = t + a < t + 2 = \cot \eta(b_{15}).$$

Both angles are in $(0, \pi/2)$, where cotangent decreases. Thus $\eta(b_{15}) < \varphi/2$, and every outer gap is at least $2\eta(b_{15})$. Either arc between two outer centers is a sum of such gaps, so their distance is at least $2b_{15} \sin \eta(b_{15}) = 2$.

The inner directions are $(2j+1)\varphi$. Comparing them with the outer directions shows that the least mixed angular separation is $\varphi - \eta(b_{15})$. It occurs at each square's two inner–outer edges, which have length 2 by (7.2). All other mixed pairs have the same radii and at least that angle, hence distance at least 2. This proves feasibility. The pentagon sides and these square edges give the two contact-angle identities in (7.6). $\square$

The constructed unit disks lie in a container of radius $1 + b_{15}$. Thus feasibility gives the upper bound

$$R_{15} \leq 1 + b_{15}. \quad (7.7)$$

To prove the matching lower bound and uniqueness, we now consider an arbitrary feasible fifteen-center set in radius b_{15} .

7.2 The number of inner centers

Set $b = b_{15}$ for this section and apply radial projection to an arbitrary feasible set of fifteen centers. We write $c = c(b)$, $\eta = \eta(b)$, $h(r) = h_b(r)$ and $f(r) = f_b(r)$ for this fixed radius. We

need the hypotheses of the angular and preliminary count bounds before excluding any count. The following finite evaluations certify them, together with the constants later needed for a single outer gap.

Proposition 7.2 (Geometric constants for fifteen centers). *The radius and boundary half-angle satisfy*

$$3.5213569 < b_{15} < 3.521357, \quad .287943 < \eta < .287944.$$

For $b_- = 3.5213569$, $c_+ = 2.386$ and $r_*^- = 1.64$, we have $2.385 < c < 2.386$ and, with $b_+ = 3.521357$, all hypotheses of Lemma 2.6 and (4.7) hold. We also have

$$11\eta > \pi, \quad 8 \arcsin(1/2.386) > \pi, \quad 2.386 - 4/2.386 < 1.$$

For $m = 10$, we have (3.9) and the strict bounds

$$\begin{aligned} 2.220 < u_{\text{gap}} < 2.221, & \quad .281 < v_{\text{gap}} < .283, \\ 1.521 < \ell_{\text{gap}} < 1.522, & \quad 0 < \omega_{10} < .551. \end{aligned} \tag{7.8}$$

Computation. We evaluate the formulas of Sections 2–3 on the exact radical b_{15} , using the outward-rounded rational arithmetic and trigonometric bounds of Appendix B. For each strict comparison, the upper endpoint on its smaller side is below the lower endpoint on its larger side. The four comparisons (2.14) are evaluated separately, as are the domain conditions and each displayed bound. These finite comparisons give all the stated inequalities; the same enclosures verify every hypothesis named in the proposition. $\square$

The outer bound (2.8) and $11\eta > \pi$ give $m \leq 10$. Apply Lemma 2.8 to the inner centers with $c_+ = 2.386$, $\nu = 8$ and $\mu = 2$. Its small-disk condition holds because $d = 2.386 - 4/2.386 < 1$, so the maximum in (2.17) is $4d^2 < 4$. It gives $k \leq 8$. Since $m + k = 15$, the remaining possibilities are $5 \leq k \leq 8$ and $7 \leq m \leq 10$.

Proposition 7.3 (Excluding six to eight inner centers). *Let $k \in \{6, 7, 8\}$ and let $\mathbf{x}_1, \dots, \mathbf{x}_k$ be centers with pairwise distances at least 2 and radii $r_i = \|\mathbf{x}_i\| \in [0, 2.386]$. Then*

$$\sum_{i=1}^k f(r_i) > \pi - (15 - k)\eta.$$

Computation. We exclude the opposite weak inequality on all possible radius vectors. For each $k \in \{6, 7, 8\}$, start with the full radial box $[0, 2.386]^k$ in inner cyclic order, using the labeling reduction (4.2). Apply radius contraction, four-corner angle bounds, pairwise angle sums, directed angle closure, and the angular-length test (4.10) with $m = 15 - k$ and the bounds in Proposition 7.2. The completed calculations examine 1819, 1149 and 81 boxes, respectively, and exclude every terminal box. The initial boxes contain every radius vector in the statement, and Section 4.2 covers every cyclic order, including ties. Proposition 4.2 therefore excludes the weak inequality for every such separated set, proving the strict inequality. $\square$

The actual projected inner set has radii less than $c < 2.386$. Lemma 2.6, whose hypotheses were certified in Proposition 7.2, gives the opposite non-strict inequality with $m = 15 - k$. Thus these three counts are impossible, and the only remaining counts are

$$k = 5, \quad m = 10. \tag{7.9}$$

7.3 One inner center in each occupied outer gap

We can now apply Lemma 3.3. The set has five inner centers by (7.9), satisfies the angular length bound, and has all the gap constants certified in Proposition 7.2. Consequently, $v_{\text{gap}} \leq r_i \leq u_{\text{gap}}$, so every inner radius belongs to $[\cdot 281, 2.221]$.

To apply the one-gap lemma, we check the six vertex-distance types of (3.12). Reflection across the horizontal axis accounts for all other pairs. Table 2 gives the squared-distance formulas and strict rational upper bounds.

Vertices	Squared distance	Strict upper bound
$\mathbf{u}_{\text{gap}}, \mathbf{v}_+$	$u_{\text{gap}}^2 + v_{\text{gap}}^2 - 2u_{\text{gap}}v_{\text{gap}} \cos \omega_{10}$	3.955
$\mathbf{u}_{\text{gap}}, \mathbf{l}_+$	$u_{\text{gap}}^2 + \ell_{\text{gap}}^2 - 2u_{\text{gap}}\ell_{\text{gap}} \cos \omega_{10}$	1.522
$\mathbf{l}_+, \mathbf{l}_-$	$4\ell_{\text{gap}}^2 \sin^2 \omega_{10}$	2.814
$\mathbf{v}_+, \mathbf{v}_-$	$4v_{\text{gap}}^2 \sin^2 \omega_{10}$	.098
$\mathbf{l}_+, \mathbf{v}_-$	$\ell_{\text{gap}}^2 + v_{\text{gap}}^2 - 2\ell_{\text{gap}}v_{\text{gap}} \cos(2\omega_{10})$	2.062
$\mathbf{l}_+, \mathbf{v}_+$	$(\ell_{\text{gap}} - v_{\text{gap}})^2$	1.541

Table 2: Vertex-distance bounds for the fifteen-center gap enclosure. All entries in the last column are exact rational strict upper bounds.

These bounds follow from (7.8) and the elementary inequalities

$$\sin \omega_{10} \leq \omega_{10}, \quad \cos \omega_{10} \geq 1 - \omega_{10}^2/2, \quad \cos(2\omega_{10}) \geq 1 - 2\omega_{10}^2.$$

For example, the first squared distance is strictly less than

$$2.221^2 + .283^2 - 2(2.220)(.281)(1 - .551^2/2) = 3.95468237582 < 3.955.$$

For the next four rows use the same upper endpoints in positive terms and lower endpoints in the subtracted positive product. For the last row use $(1.522 - .281)^2 = 1.540081 < 1.541$. Every bound is less than 4. Lemma 3.4 therefore shows that each outer gap, including its endpoint rays, contains the direction of at most one inner center.

7.4 Cyclic order and narrow radial ranges

Let q_i count outer centers between consecutive inner centers. If $q_i = 0$, those two inner directions lie in the same closed outer gap, contrary to the preceding conclusion. Therefore

$$q_i \geq 1, \quad q_1 + \cdots + q_5 = 10. \quad (7.10)$$

Placing four dividers among the nine spaces in ten objects counts $\binom{9}{4} = 126$ positive compositions. The dihedral reduction of Section 4.2 leaves sixteen representatives. The next computation checks this enumeration and covers every representative with the full radius range just proved.

Proposition 7.4 (Localization of the five inner radii). *Every projected feasible fifteen-center set in radius b has a cyclic labeling of its five inner centers for which $\mathbf{q}_{\text{out}} = (2, 2, 2, 2, 2)$ and*

$$1.693 \leq r_i \leq 1.709 \quad (1 \leq i \leq 5). \quad (7.11)$$

Computation. Generate all positive five-entry count vectors with sum 10 and retain one representative under cyclic shifts and reversal. The output is 126 vectors and 16 representatives. For each representative, begin with the smallest interval of denominator 2^{30} containing $[\cdot 281, 2.221]$ for each radius, and include all fifteen centers in the prescribed order.

Apply radius contraction, the chain-angle sum test of Lemma 3.1, and directed angle closure, using 1307 certified rational pair-angle bounds. We retain a box only when its order and all five radial intervals satisfy the conclusion. Every other box is excluded or subdivided. The completed calculation examines 63 846 boxes, accepts 175 boxes under this condition, and excludes every other terminal box. No case remains unresolved. The preceding geometric bounds place every projected feasible set in these starting ranges. The order enumeration covers every cyclic list by Section 4.2. Since every terminal box is excluded or wholly contained in one of the stated matches, Proposition 4.2 gives the conclusion for every feasible set. $\square$

The remaining argument will turn these intervals into exact equalities.

7.5 Two angle bounds near the pentagon

Each inner gap has two lower bounds: the direct distance between its inner endpoints, and the chain through its two outer centers. Increasing an inner radius near a decreases the first required angle and increases the second. We quantify both changes by Taylor estimates, then use the fact that there are five gaps to prevent cancellation around the cycle.

Let δ_i be the counterclockwise gap between inner centers i and $i + 1$, with indices modulo 5. Lemma 3.1, with $q_i = 2$, gives

$$\delta_i \geq g(r_i, r_{i+1}), \quad (7.12)$$

$$\delta_i \geq h(r_i) + 2\eta + h(r_{i+1}). \quad (7.13)$$

Since $\sum_i \delta_i = 2\pi$, these imply

$$\mathcal{L} := \sum_{i=1}^5 \max\{g(r_i, r_{i+1}), h(r_i) + h(r_{i+1}) + 2\eta\} \leq 2\pi. \quad (7.14)$$

At the construction, each entry in either maximum is 2φ by (7.6). Thus the sum is 2π there. We will show that every nonzero radial change in the allowed ranges makes this sum larger.

Measure a radial change relative to the pentagon radius by

$$\xi_i := \frac{r_i - a}{a}, \quad r_i = a(1 + \xi_i), \quad \varepsilon := 1/200.$$

The radical bounds proved in Appendix E give

$$a(1 - 1/200) < 1.693 \leq r_i \leq 1.709 < a(1 + 1/200). \quad (7.15)$$

Thus

$$|\xi_i| \leq \varepsilon \quad (1 \leq i \leq 5). \quad (7.16)$$

To use Taylor's theorem on this whole range, define the scaled angle functions

$$\Phi(\xi, \zeta) := g(a(1 + \xi), a(1 + \zeta)), \quad \Psi(\xi) := h(a(1 + \xi)). \quad (7.17)$$

Here ξ, ζ denote independent relative radial changes.

Lemma 7.5. *Put $\chi := 1/t$. For $|\xi|, |\zeta| \leq 1/200$,*

$$\Phi(\xi, \zeta) \geq 2\varphi - \chi(\xi + \zeta) - 10(\xi^2 + \zeta^2), \quad (7.18)$$

$$\Psi(\xi) + \Psi(\zeta) + 2\eta \geq 2\varphi + t(\xi + \zeta) - 10(\xi^2 + \zeta^2). \quad (7.19)$$

Proof. Proof strategy. We use the values and derivatives at zero to identify the linear terms. Uniform second-derivative bounds then control the remainder along the segment from $(0, 0)$ to (ξ, ζ) . The needed bounds follow by differentiating the cosine-rule formula; their full calculation is in Appendix E.

Equations (E.1), (E.4) and (E.6) give

$$\begin{aligned}\Phi(0, 0) &= 2\varphi, & (\Phi)_\xi(0, 0) &= (\Phi)_\zeta(0, 0) = -\chi, \\ \Psi(0) &= \varphi - \eta, & \Psi'(0) &= t,\end{aligned}\tag{7.20}$$

and, throughout the indicated square,

$$|(\Phi)_{\xi\xi}| + |(\Phi)_{\xi\zeta}| \leq 20, \quad |(\Phi)_{\zeta\zeta}| + |(\Phi)_{\xi\zeta}| \leq 20, \quad |\Psi''| \leq 20.\tag{7.21}$$

A subscript here denotes a partial derivative in the specified variable, with the other variable fixed.

For fixed ξ, ζ , set $F_{\text{seg}}(\lambda) = \Phi(\lambda\xi, \lambda\zeta)$ on $0 \leq \lambda \leq 1$, where λ is the segment parameter. The segment stays in the stated square. The chain rule and $2|\xi\zeta| \leq \xi^2 + \zeta^2$ give

$$\begin{aligned}|F''_{\text{seg}}(\lambda)| &\leq |(\Phi)_{\xi\xi}|\xi^2 + 2|(\Phi)_{\xi\zeta}||\xi\zeta| + |(\Phi)_{\zeta\zeta}|\zeta^2 \\ &\leq 20(\xi^2 + \zeta^2).\end{aligned}\tag{7.22}$$

Taylor's estimate (5.6) bounds the remainder $|F_{\text{seg}}(1) - F_{\text{seg}}(0) - F'_{\text{seg}}(0)|$ by $10(\xi^2 + \zeta^2)$. Substitution of (7.20) proves (7.18). For $\Psi(\lambda\xi) + \Psi(\lambda\zeta) + 2\eta$, the bound on Ψ'' gives the same remainder. Its constant term is 2φ and its linear term is $t(\xi + \zeta)$, proving (7.19). $\square$

7.6 The odd cycle forces equal radii

The two linear terms in Lemma 7.5 have opposite signs. Their maximum controls the absolute value of an adjacent sum of radial changes. We first show that these adjacent sums control all five changes, then compare that linear control with the quadratic Taylor error.

Lemma 7.6 (Five-cycle inequality). *For any real numbers $\xi_1, \dots, \xi_5$, with indices modulo 5,*

$$\sum_{i=1}^5 |\xi_i + \xi_{i+1}| \geq \frac{2}{5} \sum_{i=1}^5 |\xi_i|.$$

Proof. For each i , cancellation gives

$$\begin{aligned}2\xi_i &= (\xi_i + \xi_{i+1}) - (\xi_{i+1} + \xi_{i+2}) + (\xi_{i+2} + \xi_{i+3}) \\ &\quad - (\xi_{i+3} + \xi_{i+4}) + (\xi_{i+4} + \xi_i).\end{aligned}$$

The triangle inequality bounds $2|\xi_i|$ by the sum of all five adjacent absolute values. Summing for the five choices of i gives $2\sum_i |\xi_i| \leq 5\sum_i |\xi_i + \xi_{i+1}|$, as required. $\square$

Lemma 7.7. *Every projected feasible fifteen-center set in radius b_{15} has all five inner radii equal to a .*

Proof. Proof strategy. The localization result (7.11) gives $|\xi_i| \leq 1/200$. We apply the two Taylor bounds to the maximum in each gap, sum the resulting inequalities, and use Lemma 7.6 to show that any nonzero radial change would exceed the available angle 2π .

Since $t > 1$, we have $t > \chi = 1/t > 0$. For every real z , considering its two possible signs gives

$$\max\{-\chi z, tz\} \geq \chi|z|.$$

Use $z = \xi_i + \xi_{i+1}$ in Lemma 7.5. The i th maximum in (7.14) is at least

$$2\varphi + \chi|\xi_i + \xi_{i+1}| - 10(\xi_i^2 + \xi_{i+1}^2).$$

Summing makes each square occur twice, and the five constants sum to 2π . Hence

$$\mathcal{L} - 2\pi \geq \chi \sum_i |\xi_i + \xi_{i+1}| - 20 \sum_i \xi_i^2.$$

By (7.16), $\xi_i^2 \leq \varepsilon|\xi_i|$. Together with the five-cycle inequality, this yields

$$\mathcal{L} - 2\pi \geq \left(\frac{2\chi}{5} - 20\varepsilon \right) \sum_i |\xi_i|.$$

The radical identity (7.4) gives $t < 3/2$, so $\chi > 2/3$. Therefore

$$\frac{2\chi}{5} - 20\varepsilon > \frac{2}{5} \frac{2}{3} - \frac{20}{200} = \frac{1}{6}.$$

The angle bound (7.14) now implies

$$0 \geq \mathcal{L} - 2\pi \geq \frac{1}{6} \sum_i |\xi_i| \geq 0.$$

Every ξ_i is zero, so $r_i = a$ for all five inner centers. □

7.7 Equality and recovery of the original set

The five inner radii are now fixed. Their direct gap bounds (7.12) and contact angles (7.6) give $\delta_i \geq 2\varphi$. The five gaps sum to $2\pi = 10\varphi$, so each gap equals 2φ . Thus the inner centers form the regular pentagon centered at O .

Each inner gap contains exactly two outer centers by Proposition 7.4. Write its three successive angular spacings as $\delta_1^{\text{sq}}, \delta_2^{\text{sq}}, \delta_3^{\text{sq}}$. The chain bounds give

$$\begin{aligned} \delta_1^{\text{sq}} &\geq \varphi - \eta, & \delta_2^{\text{sq}} &\geq 2\eta, & \delta_3^{\text{sq}} &\geq \varphi - \eta, \\ \delta_1^{\text{sq}} + \delta_2^{\text{sq}} + \delta_3^{\text{sq}} &= 2\varphi. \end{aligned}$$

The three lower bounds already sum to 2φ , so equality holds in each. With the common outer radius b_{15} , these directions place the two outer centers at the square vertices in (7.2). This holds in every gap, proving that the projected set is an orthogonal image of $\mathcal{P}_{15}$.

Every outer center of that construction touches an inner center. The inner centers have radius $a < c$, since (7.15) gives $a < 1.711$ and the certified cutoff exceeds 2.385. They were unchanged by projection. Lemma 2.3 therefore shows that every outer center originally had radius b_{15} as well. No center moved, so every original feasible set in radius b_{15} is congruent to $\mathcal{P}_{15}$. In particular, a smaller radius is impossible. Combining this lower bound with (7.7) proves

$$R_{15} = 1 + \sqrt{6 + 2/\sqrt{5} + 4\sqrt{1 + 2/\sqrt{5}}},$$

with uniqueness up to isometry and relabeling, as claimed in Theorem 1.1.

8 Sixteen circles

Proof strategy. We first construct a packing by successive circle intersections and specify its radius exactly. For an arbitrary feasible set at that radius, the count and gap lemmas will reduce the problem to five inner centers and eleven outer centers. A finite computation will restrict their order and radii. We then bound their positions, use Lemma 5.4 to obtain exact agreement, and recover every center before projection. These steps prove both optimality and uniqueness.

8.1 Construction and its defining radius

The reference construction is symmetric about the vertical axis. Eleven centers form a boundary chain of contacts. Five inner centers are then determined successively by contacts, except that one final contact gives an equation for the radius. We use the rotation R_b , reflection S , and circle intersections $l_{\pm}$ defined in Section 6.

Let b vary in the rational interval $[b_{16}^-, b_{16}^+]$ specified in Appendix A.1; both endpoints lie strictly between 3.61542559487319 and 3.61542559487320. Define

$$\mathbf{Q}_0 = (0, -b), \quad \mathbf{Q}_j = R_b^j \mathbf{Q}_0 = (X_j, Y_j) \quad (1 \leq j \leq 5). \quad (8.1)$$

Each center has radius b , and consecutive centers have distance 2 by (6.1). The eleven boundary centers are $\mathbf{Q}_0$, the five $\mathbf{Q}_j$ for $1 \leq j \leq 5$, and their five reflections $S\mathbf{Q}_j$.

To place a symmetric inner contact pair, give the right-hand center horizontal coordinate 1. Its distance-2 contact with $\mathbf{Q}_1$ determines the higher of two intersections:

$$\mathbf{A} := \left(1, Y_1 + \sqrt{4 - (X_1 - 1)^2}\right). \quad (8.2)$$

Its reflected partner is distance 2 away. Next, place an inner center on the vertical axis below $\mathbf{Q}_5$, at distance 2 from that center and its reflection:

$$\mathbf{Z} := \left(0, Y_5 - \sqrt{4 - (X_5)^2}\right). \quad (8.3)$$

The remaining right-hand inner center must touch both $\mathbf{A}$ and $\mathbf{Z}$. Choose the intersection on the right of the directed line from the former to the latter:

$$\mathbf{M} := l_-(\mathbf{A}, \mathbf{Z}). \quad (8.4)$$

The five inner centers are $\mathbf{A}, \mathbf{M}, \mathbf{Z}, S\mathbf{M}, S\mathbf{A}$.

All these coordinates depend on b . The closing contact between $\mathbf{M}$ and $\mathbf{Q}_3$ requires

$$F_{16}(b) := \|\mathbf{M} - \mathbf{Q}_3\|^2 - 4 = 0. \quad (8.5)$$

Figure 8 distinguishes this closing contact from those already imposed by the coordinate formulas.

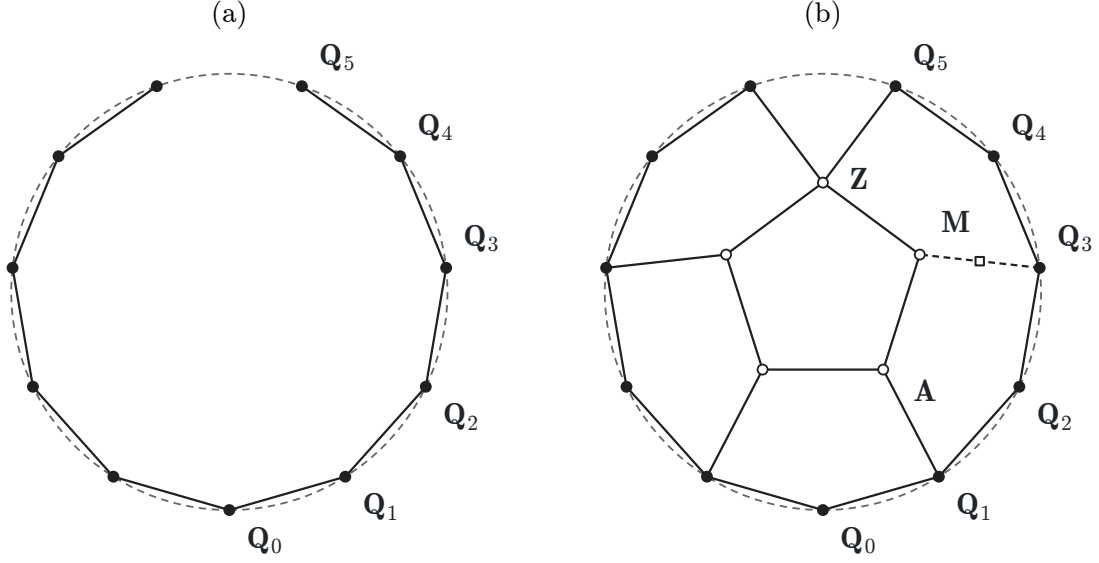


Figure 8: The sixteen-center reference construction. The outer chain consists of $Q_0, \dots, Q_5$ and the reflected centers. The inner centers are A, M, Z and the two off-axis reflections. Filled and open circles are outer and inner centers. The dashed circle has radius b . Segments are contacts of length 2; the dashed segment with a square mark is the closing contact. The marked contact between M and Q_3 gives the closing equation (8.5).

Proposition 8.1 (The sixteen-center construction). *Throughout $[b_{16}^-, b_{16}^+]$, the coordinate formulas have nonzero denominators and strictly positive radicands. The endpoint values of F_{16} have opposite signs. At every root in this interval, the stated construction is a feasible set of 16 centers, with all its proposed inner centers strictly inside $c(b)$. For every pair outside the selected contact list, $\|\mathbf{p} - \mathbf{q}\|^2 - 4 > .15861$. Hence the closing equation has at least one root in the interval.*

Computation. Proof strategy. We first verify that the coordinate formulas are defined throughout the input interval and that the closing function changes sign. We then bound all noncontact distances and inner radii. The exact contact identities and the intermediate value theorem give the feasible construction.

The input is the closed rational interval $[b_{16}^-, b_{16}^+]$ of Appendix A.1 and the coordinate formulas (8.1)–(8.5). Interval evaluation certifies that every denominator is nonzero and every radicand is strictly positive throughout this interval. The values of F_{16} at its two endpoints have opposite signs; Table 13 prints signed rational enclosures and the domain margins.

The selected pair contacts are the ten successive edges of the outer chain, the pair A, SA , and the five pairs

$$AQ_1, \quad ZQ_5, \quad MA, \quad MZ, \quad MQ_3$$

and their reflections. This is a list of 21 distinct pairs. For each of the other 99 pairs, interval evaluation certifies

$$\|\mathbf{p} - \mathbf{q}\|^2 - 4 > .15861.$$

It also certifies that all five inner radii are less than $c(b)$.

The positive domain margins make F_{16} continuous, so the opposite endpoint signs give a root by the intermediate value theorem. At a root, the closing contact is exact. Every other selected contact is exact by the preceding construction identities, and the remaining distance and containment bounds establish feasibility. $\square$

The denominator and radicand bounds make F_{16} continuous on the closed interval. The intermediate value theorem and the endpoint signs therefore give a root. Define b_{16} to be the least root in $[b_{16}^-, b_{16}^+]$; this exists because the nonempty zero set is closed in a compact interval. Evaluate the coordinates at this radius and let $\mathcal{P}_{16}$ be their sixteen-center set.

The coordinate formulas make each selected pair a contact exactly, except for the closing pair and its reflection, which follow from $F_{16}(b_{16}) = 0$. Proposition 8.1 verifies all other pair distances and the strict inner containment. The outer radii equal b_{16} by the rotation formula. Thus the whole set is feasible, proving

$$R_{16} \leq 1 + b_{16}. \quad (8.6)$$

The lower-bound proof below applies to every root in the defining interval, since all certified bounds hold throughout it.

8.2 Counts and a single outer gap

Set $b = b_{16}$ and project an arbitrary feasible sixteen-center set. We write $c = c(b)$, $\eta = \eta(b)$, $h(r) = h_b(r)$ and $f(r) = f_b(r)$ for this fixed radius. As in the preceding case, we first certify the numerical hypotheses of the common geometric bounds. The next computation also gives the five-center gap bounds for the count that will remain.

Proposition 8.2 (Geometric constants for sixteen centers). *With*

$$b_- = 3.615425, \quad b_+ = 3.615426, \quad c_+ = 2.510, \quad r_*^- = 1.7,$$

all hypotheses of Lemma 2.6 and (4.7) hold. We have

$$12\eta > \pi, \quad 8 \arcsin(1/2.510) > \pi, \quad 2.510 - 4/2.510 < 1.$$

For $m = 11$, the gap hypotheses (3.9) hold. The bounds

$$.242 < v_{\text{gap}} < .243, \quad 1.811 < u_{\text{gap}} < 1.812, \quad (8.7)$$

hold, and every squared vertex distance in (3.12) is less than 2.512607.

Computation. We evaluate the formulas of Sections 2–3 on the whole defining interval $[b_{16}^-, b_{16}^+]$, using the outward-rounded rational arithmetic and trigonometric bounds of Appendix B. For each strict comparison, the upper endpoint on its smaller side is below the lower endpoint on its larger side. The four comparisons (2.14) are evaluated separately, as are the domain conditions and each displayed bound. These finite comparisons give all the stated inequalities; the same enclosures verify every hypothesis named in the proposition. $\square$

The outer bound gives $m \leq 11$. Apply Lemma 2.8 with $c_+ = 2.510$, $\nu = 8$ and $\mu = 2$; its small-disk condition follows from $d = 2.510 - 4/2.510 < 1$. Thus $k \leq 8$. Using $m + k = 16$, we obtain $5 \leq k \leq 8$ and $8 \leq m \leq 11$.

Proposition 8.3 (Excluding six to eight inner centers). *Let $k \in \{6, 7, 8\}$ and let $\mathbf{x}_1, \dots, \mathbf{x}_k$ be centers with pairwise distances at least 2 and radii $r_i = \|\mathbf{x}_i\| \in [0, 2.510]$. Then*

$$\sum_{i=1}^k f(r_i) > \pi - (16 - k)\eta.$$

Computation. We exclude the opposite weak inequality on all possible radius vectors. For each $k \in \{6, 7, 8\}$, start with $[0, 2.510]^k$, the inner cyclic order, and the labeling inequalities (4.2). Apply radius contraction, four-corner angle bounds, angle sums, directed angle closure and (4.10) with $m = 16 - k$. The completed calculation excludes all terminal boxes for all three counts, with no retained or unresolved box. Exact counts are recorded in Appendix C. The initial boxes contain every radius vector in the statement, and Section 4.2 covers every cyclic order, including ties. Proposition 4.2 therefore excludes the weak inequality for every such separated set, proving the strict inequality. $\square$

The actual projected inner set lies in these ranges and obeys the opposite inequality by Lemma 2.6 and Proposition 8.2. Hence only

$$k = 5, \quad m = 11$$

remain. With these counts, Lemma 3.3 gives $v_{\text{gap}} \leq r_i \leq u_{\text{gap}}$, using the hypotheses certified in Proposition 8.2. The vertex bounds there are all less than 4, so Lemma 3.4 puts at most one inner center's direction in any closed outer gap.

8.3 Cyclic order and radial localization

There must now be at least one outer center between consecutive inner centers. Thus the count vector has five positive entries summing to 11. There are $\binom{10}{4} = 210$ such vectors, and dihedral equivalence leaves 26 representatives. The radial bounds (8.7) place every inner center in $[.242, 1.812]$.

Proposition 8.4 (Localization for sixteen centers). *Every projected feasible sixteen-center set in radius b has a cyclic labeling of its five inner centers, after a possible reflection, with $\mathbf{q}_{\text{out}} = (2, 2, 2, 2, 3)$. Their radii lie, in order, in the five closed intervals of radius 10^{-6} about*

$$(1.63017421, 1.72048802, 1.81155545, 1.72048802, 1.63017421). \quad (8.8)$$

Computation. Enumerate the 210 positive five-entry vectors with sum 11, reduce to their 26 dihedral representatives, and start each inner radius in the smallest interval of denominator 2^{30} containing $[.242, 1.812]$. Include all sixteen centers in the specified cyclic order. Apply radius contraction, the chain-angle sum test, and directed angle closure, using 4805 certified rational pair-angle bounds.

We retain a box only when its order is the stated vector and each whole radial interval lies in its corresponding target interval. The completed calculation examines 44 580 boxes, accepts 190 under this condition, and excludes every other terminal box. No unresolved box remains. The preceding geometric bounds place every projected feasible set in these starting ranges. The order enumeration covers every cyclic list by Section 4.2. Since every terminal box is excluded or wholly contained in one of the stated matches, Proposition 4.2 gives the conclusion for every feasible set. $\square$

Use the labeling from Proposition 8.4. To match the unknown centers to the construction, list the reference centers in cyclic order as

$$(\mathbf{A}, \mathbf{Q}_2, \mathbf{Q}_3, \mathbf{M}, \mathbf{Q}_4, \mathbf{Q}_5, \mathbf{Z}, \mathbf{SQ}_5, \mathbf{SQ}_4, \mathbf{SM}, \mathbf{SQ}_3, \mathbf{SQ}_2, \mathbf{SA}, \mathbf{SQ}_1, \mathbf{Q}_0, \mathbf{Q}_1). \quad (8.9)$$

For each reference center $\mathbf{p}$ in this list, denote the corresponding unknown center by $\mathbf{x}_{\mathbf{p}}$. This index names its matched reference position. The five inner centers are matched to $\mathbf{A}, \mathbf{M}, \mathbf{Z}, \mathbf{SM}, \mathbf{SA}$, in that order. The following computation verifies that the exact reference also has this cyclic order and these radial ranges.

8.4 A bound for every position

The unknown set and the exact reference must satisfy the same radial box and cyclic order before the directed angle bounds can be compared. We first certify those facts for the reference coordinates. We then align the boundary center matched to $\mathbf{Q}_0$ and apply the polar position estimate to the full widths of their common angular intervals.

Proposition 8.5 (Position bounds for sixteen centers). *Let $\{\mathbf{x}_p : p \in \mathcal{P}_{16}\}$ be a feasible matched set in radius b with the full cyclic order (8.9) and the closed radial intervals of (8.8). Assume every matched outer center has radius b . Align the center matched to $\mathbf{Q}_0$ with that reference point. Then*

$$\|\mathbf{x}_p - \mathbf{p}\| < 2.276 \cdot 10^{-5} < 1/10000 \quad (\mathbf{p} \in \mathcal{P}_{16}). \quad (8.10)$$

Computation. Proof strategy. We first place the reference in the same radial ranges and cyclic order as the unknown set. Common directed angle bounds then control the differences of their directions. The polar position estimate converts these angular and radial bounds to the asserted distances.

The input consists of the five closed radial intervals specified in (8.8), the full order (8.9), and the boundary interval $[b_{16}^-, b_{16}^+]$. Coordinate signs and determinants certify that the reference centers have that cyclic order, and interval norm bounds place each reference inner radius in its stated interval. The determinant test compares two directions within the same half turn; the coordinate signs specify those half turns.

For all pairs, certify four-corner angle lower bounds and apply (4.4). Use $\Pi^+ = 6908435304716/2^{40} > 2\pi$. With the reference index at $\mathbf{Q}_0$, the two directed bounds produce the widths in Table 3. Every resulting width is nonnegative. Substitution in (5.8) gives a bound strictly below $2.276 \cdot 10^{-5}$ for each center. Both the unknown and reference radii lie in the same intervals, so the radial errors are at most $2 \cdot 10^{-6}$ for inner centers and zero for outer centers. The directed bounds enclose both sets' relative directions, so their full widths bound the angular errors. Lemma 5.3 gives the claimed distances. $\square$

Reference centers	$2^{40}\varepsilon_{\theta,i}$	Strict upper bound for $\varepsilon_{\text{pos},i}$
A, SA	6 313 444	$22.760 \cdot 10^{-6}$
M, SM	2 895 618	$11.522 \cdot 10^{-6}$
Z	1 606 966	$7.285 \cdot 10^{-6}$
Outer centers other than $\mathbf{Q}_0$	1 606 966	$5.285 \cdot 10^{-6}$

Table 3: Position bounds after alignment at $\mathbf{Q}_0$. The middle column contains exact integers; the last column contains exact rational strict upper bounds. The aligned center has zero error.

Appendix D.1 gives a worked chain addition producing one row of Table 3.

8.5 The contact certificate and exact agreement

The construction has 21 pair contacts and 11 container contacts. They give $n_{\text{con}} = 32$ signed constraints. After fixing the horizontal coordinate of the center matched to $\mathbf{Q}_0$, there are $n_{\text{var}} = 31$ coordinate variables. Thus adjoining the column of ones makes the contact matrix square, with no additional columns needed.

Proposition 8.6 (Local-rigidity certificate for sixteen centers). *For the selected pair and container contacts, fix the horizontal coordinate of $\mathbf{Q}_0$. The signed linear constraints satisfy (5.14) and (5.15) with $\sigma_* = 1/250$ and $\kappa = 5$. Thus the local-rigidity threshold of Lemma 5.4 is $\sigma_*/(8\kappa)$.*

Computation. Use the reference coordinates over $[b_{16}^-, b_{16}^+]$ and the contacts of Proposition 8.1. Form the 32 by 31 signed rigidity matrix after deleting the horizontal coordinate of $\mathbf{Q}_0$, and adjoin a column of ones as in (5.18). For the supplied rational matrix $\mathbf{P}$, interval evaluation certifies

$$\|\text{Id} - \mathbf{P}\mathbf{H}\|_\infty < 10^{-10}$$

and both inequalities (5.26) with $\kappa = 5$ and $\sigma_* = 1/250$. These strict bounds hold throughout the defining interval. The rational matrix and contact-index data are specified in Appendix G. Lemma 5.6 proves invertibility and the stated inverse bound from this residual. Proposition 5.5 then supplies a normalized stress with every weight greater than σ_* and the displacement bound with constant κ . This proves the positive stress identity (5.14) and the coordinate bound (5.15). $\square$

Proposition 8.6 supplies the stress and inverse bound for Lemma 5.4, whose threshold is

$$\frac{\sigma_*}{8\kappa} = \frac{1}{10000}.$$

After alignment at $\mathbf{Q}_0$, that coordinate displacement is zero. The position bound (8.10) makes every other coordinate displacement strictly smaller than $1/10000$. All centers are feasible in radius b_{16} by assumption. Lemma 5.4 therefore forces $\mathbf{x}_p = \mathbf{p}$ for every reference center: the projected set is exactly $\mathcal{P}_{16}$.

8.6 The lower bound and recovery before projection

The center $\mathbf{Q}_1$ touches the inner center $\mathbf{A}$. Lemma 2.3 puts its original radius at b_{16} , so no smaller center disk can contain sixteen centers. Together with (8.6), this proves

$$R_{16} = 1 + b_{16}.$$

To obtain uniqueness, it remains to show that projection did not change any other center.

The outer centers $\mathbf{Q}_1, \mathbf{Q}_3, \mathbf{Q}_5$ touch $\mathbf{A}, \mathbf{M}, \mathbf{Z}$, respectively. Their reflected partners have the corresponding contacts. All six outer centers therefore originally had radius b_{16} by Lemma 2.3. Each remaining outer center is adjacent in the boundary chain to one of these six. Lemma 2.4 then makes its original radius either c or b_{16} . We exclude the cutoff alternative with the following finite distance bounds.

Proposition 8.7 (Excluding cutoff positions for sixteen centers). *For every $b \in [b_{16}^-, b_{16}^+]$, the construction coordinates satisfy*

$$\begin{aligned} \|(c(b)/b)\mathbf{Q}_0 - \mathbf{A}\|^2 &< 2.4924, \\ \|(c(b)/b)\mathbf{Q}_2 - \mathbf{A}\|^2 &< 1.6250, \\ \|(c(b)/b)\mathbf{Q}_4 - \mathbf{M}\|^2 &< 1.0159. \end{aligned} \tag{8.11}$$

The same bounds hold for the reflected pairs.

Computation. Evaluate the coordinate formulas on the whole closed interval $[b_{16}^-, b_{16}^+]$, using the certified domain margins in Table 13. Form each displayed difference, square its two coordinates and add. Outward-rounded interval evaluation gives an upper endpoint strictly below the stated rational bound. Reflection preserves squared distance, proving the remaining cases. $\square$

Every bound in (8.11) is less than 4. The inner contact witnesses were unchanged by projection, so an original center at any of these cutoff positions would violate separation. Every original outer radius is therefore b_{16} . The original set coincides with its projection and hence

with the reference after the chosen isometry and relabeling. This proves the sixteen-circle uniqueness assertion in Theorem 1.1.

The closing equation also has only one root in its defining interval. Indeed, if two roots were distinct, Proposition 8.1 would give a feasible construction at the smaller one, whereas the argument just completed would give a lower bound equal to the larger one. All its interval estimates hold for either root, so these two conclusions would contradict each other.

9 Seventeen circles

Proof strategy. The construction gives an upper bound and a reference for the local argument. For an arbitrary feasible set at that radius, the initial exclusions will leave either eleven or ten outer centers. The next computation will exclude eleven and restrict the ten-outer-center case to a small neighborhood of the reference. We then apply Lemma 5.4 and recover every original center through its contacts. Two reference inner centers share a direction, so all order reductions must allow a tie there.

9.1 Construction and its defining radius

We build a symmetric configuration from its lower contacts, then extend it upward on both sides. All pair contacts will follow from circle intersections and equilateral triangles. The final pair of centers must also lie on the boundary; that condition determines the radius.

Let b vary over

$$\begin{aligned} b_{17}^- &= 3.7920337483105791701491239532, \\ b_{17}^+ &= 3.7920337483105791701491239534. \end{aligned} \tag{9.1}$$

Use $\tau_b = \sqrt{b^2 - 1}$ and the operators R_b , J and S from (6.1) and (6.2). Begin with $\mathbf{A} = (1, -\tau_b)$ and its reflection. Both have radius b , and their distance is 2. The upper vertex of their equilateral triangle is

$$\mathbf{F} := (0, -\tau_b + \sqrt{3}).$$

The next boundary center on the right and its adjacent inner center are

$$\begin{aligned} \mathbf{C} &:= R_b \mathbf{A} = (3 - 4/b^2, -\tau_b + 4\tau_b/b^2), \\ \mathbf{E} &:= \mathbf{C} + (-1, \sqrt{3}). \end{aligned}$$

The last vector has length 2, giving the contact between $\mathbf{C}$ and $\mathbf{E}$. Also $\mathbf{E} - \mathbf{F} = (2 - 4/b^2, 4\tau_b/b^2)$, whose squared length is

$$(2 - 4/b^2)^2 + (4\tau_b/b^2)^2 = 4 - 16/b^2 + 16/b^4 + 16(b^2 - 1)/b^4 = 4.$$

Thus $\mathbf{E}$ touches $\mathbf{F}$ as well.

Reflect $\mathbf{F}$ across the horizontal line through $\mathbf{E}$. The vertical difference is $4\tau_b/b^2$, so the resulting axial center is

$$\mathbf{G} := (0, -\tau_b + \sqrt{3} + 8\tau_b/b^2).$$

This reflection preserves its distance 2 from $\mathbf{E}$. Reflection in the vertical axis supplies all corresponding left-hand contacts. Both $\mathbf{F}$ and $\mathbf{G}$ lie on that axis.

Next place a boundary center $\mathbf{H}$ at distance 2 from $\mathbf{E}$. To solve these two circle equations, put $e := \|\mathbf{E}\|^2$ and write $\mathbf{H} = \alpha_E \mathbf{E} + \beta_E J\mathbf{E}$. The two summands are perpendicular. Subtracting the equations $\|\mathbf{H}\|^2 = b^2$ and $\|\mathbf{H} - \mathbf{E}\|^2 = 4$ gives $2\mathbf{H} \cdot \mathbf{E} = b^2 + e - 4$. Consequently the chosen intersection is

$$\alpha_E := \frac{b^2 + e - 4}{2e}, \quad \beta_E := \sqrt{\frac{b^2}{e} - \alpha_E^2}, \quad \mathbf{H} := \alpha_E \mathbf{E} + \beta_E J\mathbf{E}. \tag{9.2}$$

The positive coefficient of the quarter-turn chooses the intersection on the counterclockwise side of the direction of $\mathbf{E}$. Put $\mathbf{M} := \mathbf{R}_b \mathbf{H}$; it is another boundary center and touches $\mathbf{H}$.

The remaining three right-hand or axial centers are determined by contacts with those already constructed:

$$\mathbf{J} := \mathbf{l}_+(\mathbf{G}, \mathbf{M}), \quad (9.3)$$

$$\mathbf{K} := \left(0, J_y + \sqrt{4 - (J_x)^2}\right), \quad (9.4)$$

$$\mathbf{L} := \frac{\mathbf{J} + \mathbf{K}}{2} + \frac{\sqrt{3}}{2} \mathbf{J}(\mathbf{J} - \mathbf{K}). \quad (9.5)$$

Here (J_x, J_y) are the coordinates of $\mathbf{J}$. The first formula gives its distance-2 contacts with $\mathbf{G}$ and $\mathbf{M}$; the second places $\mathbf{K}$ above it on the axis at distance 2. The last formula constructs an equilateral triangle on their segment, so $\mathbf{L}$ touches both. The boundary condition for this last center is the closing equation

$$F_{17}(b) := \|\mathbf{L}\|^2 - b^2 = 0. \quad (9.6)$$

Figure 9 shows the successive contacts and the two inner centers on the same upward ray.

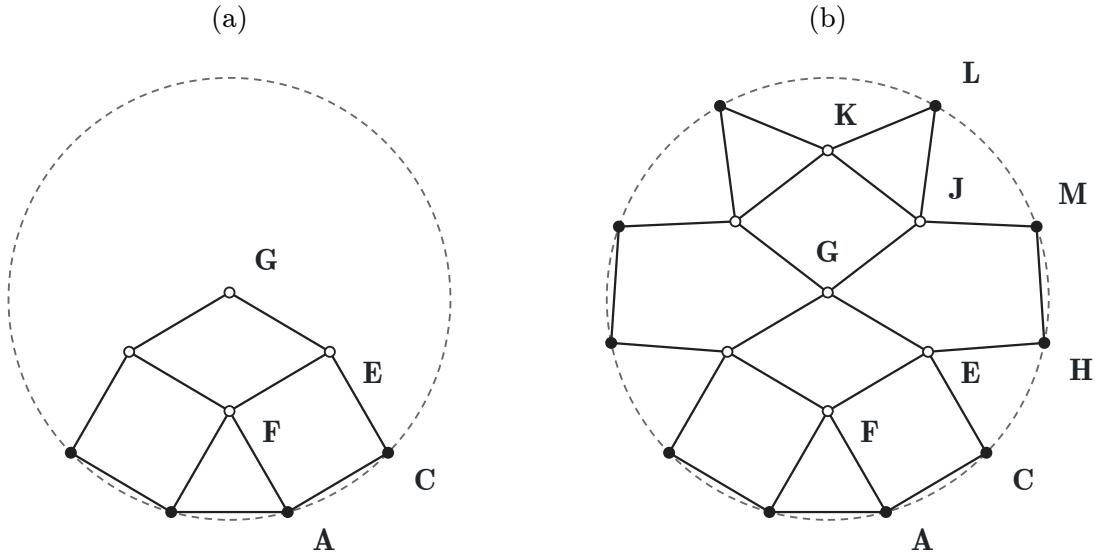


Figure 9: The seventeen-center construction. The first panel shows $\mathbf{A}, \mathbf{C}, \mathbf{F}, \mathbf{E}, \mathbf{G}$ and their reflections. The second adds $\mathbf{H}, \mathbf{M}, \mathbf{J}, \mathbf{K}, \mathbf{L}$ and their reflections. Filled and open circles are outer and inner centers, and the dashed circle has radius b . Solid segments represent distance-2 contacts. The final equation puts $\mathbf{L}$ on radius b . The centers $\mathbf{G}, \mathbf{K}$ share the upward ray.

The contact list consists of the pair $\mathbf{A}, \mathbf{SA}$ and the following twelve pairs, together with their reflections:

$$\begin{aligned} &\mathbf{AC}, \quad \mathbf{AF}, \quad \mathbf{CE}, \quad \mathbf{FE}, \quad \mathbf{EH}, \quad \mathbf{EG}, \\ &\mathbf{HM}, \quad \mathbf{GJ}, \quad \mathbf{MJ}, \quad \mathbf{JK}, \quad \mathbf{JL}, \quad \mathbf{KL}. \end{aligned} \quad (9.7)$$

Every listed distance is exactly 2 by the construction steps above. There are 25 distinct contact pairs.

Proposition 9.1 (The seventeen-center construction). *Throughout $[b_{17}^-, b_{17}^+]$, the coordinate formulas have nonzero denominators and strictly positive radicands. The endpoint values of F_{17} have opposite signs. At every root in this interval, the stated construction is a feasible set of 17 centers, with all its proposed inner centers strictly inside $c(b)$. For every pair outside the*

selected contact list, $\|\mathbf{p} - \mathbf{q}\|^2 - 4 > .1412526818$. Each proposed inner center also satisfies $b^2 - \|\mathbf{p}\|^2 > 7.8901595432$. The derivative satisfies $F'_{17}(b) < 0$. Hence the closing equation has exactly one root in the interval.

Computation. Proof strategy. We first verify that the coordinate formulas are defined throughout the input interval and that the closing function changes sign. We then bound all noncontact distances and inner radii. The exact contact identities and the intermediate value theorem give the feasible construction.

Evaluate the coordinate formulas and F_{17} over $[b_{17}^-, b_{17}^+]$. Every denominator is nonzero and every radicand is strictly positive. The endpoint values of F_{17} have opposite signs, and $F'_{17} < 0$ throughout the interval. Table 13 prints rational enclosures for these values and the domain margins.

The proposed outer centers are **A, C, H, M, L** and their reflections. The seven proposed inner centers are **F, E, SE, G, J, SJ, K**. For all 111 pairs outside (9.7) and the additional contact **A, SA**, certify

$$\|\mathbf{p} - \mathbf{q}\|^2 - 4 > .1412526818.$$

For each proposed inner center, certify $b^2 - \|\mathbf{p}\|^2 > 7.8901595432$ and $\|\mathbf{p}\| < c(b)$. Rational interval evaluation gives these strict bounds. The positive domain margins make F_{17} continuous, so the opposite endpoint signs give a root by the intermediate value theorem. At a root, the closing contact is exact. Every other selected contact is exact by the preceding construction identities, and the remaining distance and containment bounds establish feasibility. The strict derivative sign makes the root unique. $\square$

The strict domain bounds make F_{17} continuously differentiable on the defining interval. The endpoint signs give a root, and its strictly negative derivative makes that root unique. Call it b_{17} and let $\mathcal{P}_{17}$ be the seventeen listed centers evaluated there. Eight outer radii equal b_{17} by the coordinate formulas; the other two equal it by $F_{17}(b_{17}) = 0$. The exact contact equalities, the noncontact bounds and the inner containment prove feasibility. Thus

$$R_{17} \leq 1 + b_{17}. \quad (9.8)$$

9.2 Restricting the numbers and radii

Set $b = b_{17}$ and radially project an arbitrary feasible seventeen-center set. We write $c = c(b)$, $\eta = \eta(b)$, $h(r) = h_b(r)$ and $f(r) = f_b(r)$ for this fixed radius. The following calculation certifies the common geometric hypotheses and an additional radius bound needed when there are eleven outer centers.

Proposition 9.2 (Geometric constants for seventeen centers). *For*

$$b_- = 3.792033, \quad b_+ = 3.792034, \quad c_+ = 2.738, \quad r_*^- = 1.9,$$

all hypotheses of Lemma 2.6 and (4.7) hold. Moreover,

$$12\eta > \pi, \quad 9 \arcsin(1/2.738) > \pi, \quad (2.738 - 4/2.738)^2 < 2.$$

For the eleven-outer-center case, set

$$\omega_{11}^+ := 3.141592653589793239 - 10(.266867098322829).$$

We have $\eta > .266867098322829$ and

$$0 < \omega_{11} < \omega_{11}^+ < \pi/2, \quad c < b_{17}^- \cos \omega_{11}^+, \quad D(2.368, b_{17}^+, \omega_{11}^+) < 4. \quad (9.9)$$

Computation. We evaluate the formulas of Sections 2–3 on the whole defining interval $[b_{17}^-, b_{17}^+]$, using the outward-rounded rational arithmetic and trigonometric bounds of Appendix B. For each strict comparison, the upper endpoint on its smaller side is below the lower endpoint on its larger side. The four comparisons (2.14) are evaluated separately, as are the domain conditions and each displayed bound. These finite comparisons give all the stated inequalities; the same enclosures verify every hypothesis named in the proposition. $\square$

The outer angle bound gives $m \leq 11$. In Lemma 2.8, use $c_+ = 2.738$, $\nu = 9$ and $\mu = 4$. For $d = 2.738 - 4/2.738$, the small-disk maximum is $2d^2 < 4$, so $k \leq 9 + 4 - 2 = 11$. Consequently $6 \leq k \leq 11$ and $6 \leq m \leq 11$.

Proposition 9.3 (Excluding eight to eleven inner centers). *Let $k \in \{8, 9, 10, 11\}$ and let $\mathbf{x}_1, \dots, \mathbf{x}_k$ be centers with pairwise distances at least 2 and radii $r_i = \|\mathbf{x}_i\| \in [0, 2.738]$. Then*

$$\sum_{i=1}^k f(r_i) > \pi - (17 - k)\eta.$$

Computation. We exclude the opposite weak inequality on all possible radius vectors. For each $k \in \{8, 9, 10, 11\}$, start with $[0, 2.738]^k$, inner cyclic order, and the labeling inequalities (4.2). Apply radius contraction, four-corner angle bounds, angle sums, directed angle closure and (4.10) with $m = 17 - k$. The completed calculations examine 100 475, 4585, 1267 and 777 boxes, respectively, and exclude all terminal boxes without an unresolved case. The initial boxes contain every radius vector in the statement, and Section 4.2 covers every cyclic order, including ties. Proposition 4.2 therefore excludes the weak inequality for every such separated set, proving the strict inequality. $\square$

The actual inner set lies in that range and obeys the opposite non-strict inequality by Lemma 2.6, with the hypotheses certified in Proposition 9.2. Thus the possible counts reduce to

$$(m, k) = (11, 6) \quad \text{or} \quad (10, 7). \quad (9.10)$$

When $m = 11$, apply Lemma 3.2 with $r_{\text{cap}} = 2.368$, the defining endpoints $b_{17}^\pm$, and (9.9). It gives $r_i < 2.368$ for every inner center. For $m = 10$, the common cutoff bound still gives $r_i < 2.738$. These are the starting ranges for the next computation.

9.3 Orders, including equal directions

Use every nonnegative count vector satisfying

$$q_i \geq 0, \quad \sum_i q_i = m.$$

This includes consecutive inner centers with no outer center between them. The order convention in Section 4.2 also includes equal directions and a freely assigned direction at O . In the reference set, $\mathbf{G}$ and $\mathbf{K}$ lie on the same upward ray, so both orders of those two centers must be covered.

Adding 1 to each entry reduces enumeration to positive compositions. Thus there are $\binom{m+k-1}{k-1}$ count vectors: 4368 for $(m, k) = (11, 6)$ and 8008 for $(10, 7)$. Dihedral reduction gives 392 and 600 representatives. The following computation checks those counts as part of its enumeration.

Proposition 9.4 (Exclusion and localization for seventeen centers). *Every projected feasible seventeen-center set in radius b has ten outer and seven inner centers. The inner centers admit a cyclic labeling with outer counts*

$$(q_0, \dots, q_6) = (0, 1, 2, 2, 2, 2, 1), \quad (9.11)$$

and rational radial targets

$$(t_0, \dots, t_6) := (.10925499699, 2.54742230599, 2.06840968290, 1.94669000932, 1.92575180443, 1.94669000932, 2.06840968290). \quad (9.12)$$

Each radius r_i is within 10^{-7} of t_i , or each is within 10^{-7} of t_{1-i} ; indices in the second alternative are read modulo 7. The weak bounds include both endpoints.

Computation. Proof strategy. We enumerate every permitted cyclic order and cover its radial ranges by closed boxes. Valid inequalities exclude boxes; the remaining boxes must lie wholly in one of the stated matches. Complete resolution of these boxes gives the proposition by Proposition 4.2.

For $(m, k) = (11, 6)$, enumerate all 392 representative orders and start each radius in the smallest interval of denominator 2^{40} containing $[0, 2.368]$. Apply radius contraction, chain-angle sums and directed angle closure, using 2353 certified rational angle bounds. The calculation examines 107 250 boxes and excludes all terminal boxes.

For $(m, k) = (10, 7)$, use all 600 representatives and initial interval $[0, 2.738]$, rounded outward on the same grid. The same tests use 159 816 certified angle bounds. We retain a box only when a cyclic choice of labels gives the stated order and every whole radial interval lies in the corresponding target interval, with one common choice of the two target orientations. The completed calculation examines 15 297 824 boxes, accepts 8423 under these conditions, and excludes every other terminal box. Both calculations finish without unresolved cases. The preceding geometric bounds place every projected feasible set in these starting ranges. The order enumeration covers every cyclic list by Section 4.2. Since every terminal box is excluded or wholly contained in one of the stated matches, Proposition 4.2 gives the conclusion for every feasible set. $\square$

The first two reference directions coincide, but their target radii differ. To interpret the reversed target match, starting the reversed list at old index 1 changes targets and following counts by the rule

$$(t'_i, q'_i) = (t_{1-i}, q_{-i}),$$

with indices read modulo 7. For (9.11), $q_{-i} = q_i$. In (9.12), the reversed sequence exchanges the first two entries and leaves the numerical sequence of the other five unchanged. Thus choosing the appropriate reading direction makes the target list the displayed one, including either possible order at the tie.

Match the unknown centers to the reference list

$$(\mathbf{G}, \mathbf{K}, \mathbf{SL}, \mathbf{SJ}, \mathbf{SM}, \mathbf{SH}, \mathbf{SE}, \mathbf{SC}, \mathbf{SA}, \mathbf{F}, \mathbf{A}, \mathbf{C}, \mathbf{E}, \mathbf{H}, \mathbf{M}, \mathbf{J}, \mathbf{L}). \quad (9.13)$$

For a reference point $\mathbf{p}$, write $\mathbf{x}_\mathbf{p}$ for its matched unknown center. In particular, the inner target order is

$$(\mathbf{G}, \mathbf{K}, \mathbf{SJ}, \mathbf{SE}, \mathbf{F}, \mathbf{E}, \mathbf{J}).$$

The reference labels specify a correspondence; every subsequent bound applies separately to each unknown center.

9.4 Position bounds and exact agreement

We align the center matched to $\mathbf{SA}$ with that reference center. Both are outer, so the alignment fixes their positions. We will certify the common order and radial box, use the directed angle widths to bound all other positions, and compare those bounds with the threshold in Lemma 5.4.

Proposition 9.5 (Position bounds for seventeen centers). *Let $\{\mathbf{x}_{\mathbf{p}} : \mathbf{p} \in \mathcal{P}_{17}\}$ be a feasible matched set in radius b with the full cyclic order (9.13) and the closed radial intervals of (9.12). Assume every matched outer center has radius b . Align the center matched to $\mathbf{SA}$ with that reference point. Then*

$$\|\mathbf{x}_{\mathbf{p}} - \mathbf{p}\| < 7.562 \cdot 10^{-6} < 1/16000 \quad (\mathbf{p} \in \mathcal{P}_{17}). \quad (9.14)$$

Computation. Proof strategy. We first place the reference in the same radial ranges and cyclic order as the unknown set. Common directed angle bounds then control the differences of their directions. The polar position estimate converts these angular and radial bounds to the asserted distances.

Use the target intervals in (9.12), the full order (9.13), and the defining boundary interval. Interval norms place the reference radii in the targets. Coordinate signs and pair determinants certify the reference cyclic order. For the sole tied pair $\mathbf{G}, \mathbf{K}$, both horizontal coordinates are exactly zero and both vertical coordinates are positive, which certifies the permitted order at equal directions.

For every pair, certify four-corner angle lower bounds and apply the directed additions (4.4), with $\Pi^+ = 6908435304716/2^{40}$. Relative to the aligned reference center, the resulting widths are those in Table 4. Every resulting width is nonnegative. Substitution in (5.8) gives a bound strictly below $7.562 \cdot 10^{-6}$ for each center. Both the unknown and reference radii lie in the same intervals, so the radial errors are at most $2 \cdot 10^{-7}$ for inner centers and zero for outer centers. The directed bounds enclose both sets' relative directions, so their full widths bound the angular errors. Lemma 5.3 gives the claimed distances. $\square$

Reference centers	$2^{40}\varepsilon_{\theta,i}$	Strict upper bound for $\varepsilon_{\text{pos},i}$
G	2 134 475	$7.562 \cdot 10^{-6}$
F	642 620	$2.417 \cdot 10^{-6}$
K, E, SE, J, SJ	564 969	$2.149 \cdot 10^{-6}$
L, SL	595 888	$2.056 \cdot 10^{-6}$
Other outer centers except SA	564 969	$1.949 \cdot 10^{-6}$

Table 4: Certified position bounds for seventeen centers. The middle column contains exact integers; the last column contains exact rational strict upper bounds. The aligned center has zero displacement.

This bounds the two centers matched to the tied reference directions separately. Appendix D.2 gives the chain additions for the largest bound, that of $\mathbf{G}$.

Proposition 9.6 (Local-rigidity certificate for seventeen centers). *For the selected pair and container contacts, fix the horizontal coordinate of $\mathbf{SA}$. The signed linear constraints satisfy (5.14) and (5.15) with $\sigma_* = 1/250$ and $\kappa = 8$. Thus the local-rigidity threshold of Lemma 5.4 is $\sigma_*/(8\kappa)$.*

Computation. Use the 25 pair contacts in the construction and the ten boundary contacts. Delete the horizontal coordinate of $\mathbf{SA}$, giving a 35 by 33 signed rigidity matrix. Adjoin a column of ones and the one rational column specified in Appendix G. For the supplied rational matrix $\mathbf{P}$, certify a residual norm less than 10^{-10} and both inequalities (5.26) with

$$\kappa = 8, \quad \sigma_* = 1/250, \quad \frac{\sigma_*}{8\kappa} = 1/16000.$$

Exact interval bounds verify these inequalities throughout the defining interval. Lemma 5.6 proves invertibility and the stated inverse bound from this residual. Proposition 5.5 then supplies a normalized stress with every weight greater than σ_* and the displacement bound with constant κ . This proves the positive stress identity (5.14) and the coordinate bound (5.15). $\square$

Proposition 9.6 supplies the positive stress and inverse bound required by Lemma 5.4. Alignment gives the zero horizontal displacement, and (9.14) bounds every coordinate displacement strictly below its threshold. The unknown set satisfies every selected separation and containment inequality by feasibility. Thus Lemma 5.4 gives $\mathbf{x}_p = \mathbf{p}$ for every matched center. The projected set is exactly the reference.

9.5 The lower bound and uniqueness

The outer center $\mathbf{A}$ touches the inner center $\mathbf{F}$. By Lemma 2.3, its original radius was b_{17} , so no smaller center disk is possible. Together with (9.8), this proves

$$R_{17} = 1 + b_{17}.$$

For uniqueness, we apply the same argument to every outer center. On the right, $\mathbf{A}, \mathbf{C}, \mathbf{H}, \mathbf{M}, \mathbf{L}$ touch, respectively, $\mathbf{F}, \mathbf{E}, \mathbf{E}, \mathbf{J}, \mathbf{J}$. The reflected contacts cover the left side. The witnesses are all inner centers with radii below c , by Proposition 9.1. Lemma 2.3 therefore fixes every original outer radius at b_{17} . No center moved under projection, proving uniqueness up to isometry and relabeling as stated in Theorem 1.1.

10 Eighteen circles

Proof strategy. The hexagon–dodecagon construction gives radius $\sqrt{2} + \sqrt{6}$. For the lower bound, twelve outer centers already force a contact and hence an original boundary center. Finite exclusions will reduce the other possibilities to ten or eleven outer centers and place their radii close to six specified reference sets. Contact-angle bounds then control their positions, and local rigidity either excludes the case or again forces an original boundary center. After proving the lower bound, we exhibit the continuous family stated in Theorem 1.1.

10.1 A hexagon and a dodecagon

Set

$$b_{18} := \sqrt{2} + \sqrt{6} = 4 \cos(\pi/12) = \csc(\pi/12). \quad (10.1)$$

The cosine difference formula gives $\cos(\pi/12) = (\sqrt{6} + \sqrt{2})/4$. The last identity follows from $2 \sin(\pi/12) \cos(\pi/12) = \sin(\pi/6) = 1/2$. The reference construction for the upper bound is

$$\mathcal{P}_{18} := \{b_{18}\mathbf{e}(j\pi/6) : 0 \leq j \leq 11\} \cup \{2\mathbf{e}(\pi/12 + j\pi/3) : 0 \leq j \leq 5\}.$$

It consists of a regular dodecagon on the boundary and a regular hexagon of radius 2, with its directions midway between alternate boundary directions.

The shortest boundary chord has length $2b_{18} \sin(\pi/12) = 2$. The inner hexagon also has side length 2. A mixed pair has angular separation at least $\pi/12$, and hence squared distance at least

$$b_{18}^2 + 4 - 4b_{18} \cos(\pi/12) = 4. \quad (10.2)$$

These checks cover every pair and prove feasibility. Thus

$$R_{18} \leq 1 + \sqrt{2} + \sqrt{6}. \quad (10.3)$$

Figure 10 displays the two polygons and their contacts. Adding O gives a nineteen-center packing at the same radius, since O has distance 2 from each hexagon vertex and b_{18} from each outer center. The lower bound below considers arbitrary eighteen-center sets, including sets with no center at O .

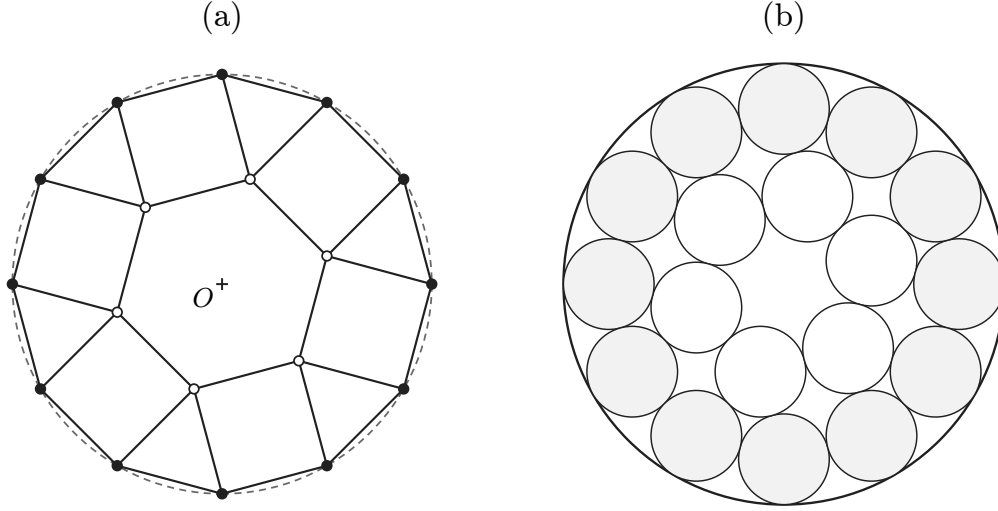


Figure 10: The eighteen-center construction $\mathcal{P}_{18}$. Six centers form a regular hexagon of radius 2; twelve lie on radius b_{18} . Their direction offset is $\pi/12$. In the first panel, open circles mark the hexagon, filled circles mark the dodecagon, and the dashed circle has radius b_{18} . Segments indicate contacts. The second panel shows the corresponding unit disks.

10.2 Reduction to three outer counts

Set $b = b_{18}$ and project an arbitrary feasible eighteen-center set. We write $c = c(b)$, $\eta = \eta(b)$, $h(r) = h_b(r)$ and $f(r) = f_b(r)$ for this fixed radius. The exact radius gives

$$\eta = \pi/12, \quad c = 2\sqrt{2}.$$

The outer bound (2.8) therefore yields $m \leq 12$. We certify the remaining geometric conditions needed to bound the inner count and use the angular length inequality.

Proposition 10.1 (Geometric constants for eighteen centers). *With the exact rational bounds*

$$b_- = 3.863703305, \quad b_+ = 3.863703306, \quad c_+ = 2.828428, \quad r_*^- = 1.9999999,$$

the hypotheses of Lemma 2.6 and (4.7) hold. In addition,

$$9 \arcsin(1/2.828428) > \pi, \quad 2.828428 - 4/2.828428 < 3/2,$$

and $2(3/2)^2(1 - \cos(2\pi/5)) < 4$.

Computation. We evaluate the formulas of Sections 2–3 on the exact radical b_{18} , using the outward-rounded rational arithmetic and trigonometric bounds of Appendix B. For each strict comparison, the upper endpoint on its smaller side is below the lower endpoint on its larger side. The four comparisons (2.14) are evaluated separately, as are the domain conditions and each displayed bound. These finite comparisons give all the stated inequalities; the same enclosures verify every hypothesis named in the proposition. $\square$

Apply Lemma 2.8 with $c_+ = 2.828428$, $\nu = 9$ and $\mu = 5$. Its small-disk condition holds because $d < 3/2$, $d^2 < 9/4 < 4$, and the last comparison in Proposition 10.1 bounds the other term. Thus $k \leq 12$. With $m + k = 18$ and $m \leq 12$, we obtain $6 \leq k \leq 12$.

Proposition 10.2 (Excluding nine to twelve inner centers). *Let $k \in \{9, 10, 11, 12\}$ and let $\mathbf{x}_1, \dots, \mathbf{x}_k$ be centers with pairwise distances at least 2 and radii $r_i = \|\mathbf{x}_i\| \in [0, 2.828428]$. Then*

$$\sum_{i=1}^k f(r_i) > \pi - (18 - k)\eta.$$

Computation. We exclude the opposite weak inequality on all possible radius vectors. For each $k \in \{9, 10, 11, 12\}$, start with $[0, 2.828428]^k$, inner cyclic order, and (4.2). Apply the radius, pair-angle, angle-sum and directed-closure tests of Section 4, together with (4.10) for $m = 18 - k$. Every terminal box is excluded for all four counts, with no unresolved box; Appendix C records the exact counts. The initial boxes contain every radius vector in the statement, and Section 4.2 covers every cyclic order, including ties. Proposition 4.2 therefore excludes the weak inequality for every such separated set, proving the strict inequality. $\square$

The actual projected set has inner radii below $c < 2.828428$ and satisfies the reverse weak inequality by Lemma 2.6. Thus only

$$(m, k) = (12, 6), \quad (11, 7), \quad (10, 8) \quad (10.4)$$

remain. When $m = 12$, all twelve outer gaps are at least $\pi/6$ and sum to 2π , so each is exactly $\pi/6$. Consecutive outer centers touch. Lemma 2.3 then forces at least one original radius to equal b_{18} . This proves the lower bound for that count. We now treat the other two counts.

10.3 Six reference sets defined by contacts

The remaining search will compare radii with 0, 2 and $c = 2\sqrt{2}$. Radius 2 is the contact radius with a center at O . The cutoff radius is useful for a different reason: if local rigidity forces an unknown inner center to have exactly that radius, it contradicts the strict inner bound $r < c$. Thus some references will exclude a case by exact agreement.

The required contact angles at these radii are

$$g(b_{18}, b_{18}) = \pi/6, \quad h(2) = \pi/12, \quad h(c) = \pi/6, \quad g(2, 2) = \pi/3.$$

The first two follow from (10.1) and (10.2); the third is (2.11). The last is the angle in an equilateral triangle with two vertices at radius 2.

We construct six reference sets by specifying their noncentral interior radii and the number q_i of boundary centers following each such center. For $q_i \geq 1$, choose each counterclockwise step along the intervening chain to equal its required contact angle. The next interior direction is then separated by

$$\delta_i^* = h(r_i^*) + (q_i - 1)\pi/6 + h(r_{i+1}^*). \quad (10.5)$$

If the current direction is θ_i^* , the intervening boundary directions are $\theta_i^* + h(r_i^*) + j\pi/6$ for $0 \leq j < q_i$. In the one instance with $q_i = 0$, both endpoint radii are 2 and we use their contact gap $\pi/3$. This specifies every direction from the first one.

The first five rows of Table 5 have positive counts. If n_2 is their number of radius-2 centers and n_c their number of cutoff-radius centers, summing (10.5) gives

$$\sum_i \delta_i^* = 2(\pi n_2/12 + \pi n_c/6) + (m - n_2 - n_c)\pi/6 = (m + n_c)\pi/6. \quad (10.6)$$

Thus ten boundary centers need two cutoff-radius centers to close one turn, and eleven need one. In the sixth row, the six radius-2 centers have six gaps of $\pi/3$: one direct contact gap and five gaps containing two boundary centers each. That row closes as well.

To illustrate the construction rule, $\mathcal{Q}_3$ starts at radius c and direction $\pi/6$, followed by five radius-2 centers. Its first gap is $\pi/6 + \pi/12 = \pi/4$, its next four are $\pi/3$, and its last is $\pi/12 + \pi/6 + \pi/6 = 5\pi/12$. These sum to 2π . The noncentral interior directions are

$$\pi/6, \quad 5\pi/12, \quad 3\pi/4, \quad 13\pi/12, \quad 17\pi/12, \quad 7\pi/4.$$

Reference	First direction	Noncentral interior radii	Following counts
$\mathcal{Q}_1$	$\pi/3$	$c, 2, c, 2, 2, 2$	$1, 1, 1, 2, 2, 2, 1$
$\mathcal{Q}_2$	$5\pi/3$	$c, 2, 2, c, 2, 2$	$1, 2, 1, 1, 2, 2, 1$
$\mathcal{Q}_3$	$\pi/6$	$c, 2, 2, 2, 2, 2$	$1, 2, 2, 2, 2, 2$
$\mathcal{Q}_4$	$\pi/2$	$c, 2, 2, 2, 2, 2$	$1, 2, 2, 2, 3, 1$
$\mathcal{Q}_5$	$5\pi/6$	$c, 2, 2, 2, 2, 2$	$1, 2, 2, 3, 2, 1$
$\mathcal{Q}_6$	$\pi/12$	$2, 2, 2, 2, 2, 2$	$0, 2, 2, 2, 2, 2$

Table 5: Exact data for the six references. In this table $c = 2\sqrt{2}$. Each row determines directions by (10.5), and O is included in every set. The first five sets have eighteen centers; the sixth has seventeen.

The boundary directions are every multiple of $\pi/6$ except $\pi/6$ itself. This example shows how the row determines all centers without an approximate coordinate construction.

More generally, put $\Omega := \{j\pi/6 : 0 \leq j \leq 11\}$. The boundary directions in the six rows are obtained from Ω by omitting, respectively,

$$\{\pi/3, 5\pi/6\}, \quad \{5\pi/3, \pi/2\}, \quad \{\pi/6\}, \quad \{\pi/2\}, \quad \{5\pi/6\}, \quad \{\pi/6, \pi/3\}.$$

In particular every row contains the boundary center $(0, -b_{18})$, which will be the alignment point in the local argument.

The coordinate entries belong to the number field $\mathbb{Q}(\sqrt{2}, \sqrt{3})$: they have the form $a_0 + a_2\sqrt{2} + a_3\sqrt{3} + a_6\sqrt{6}$ with rational coefficients. Arithmetic on the four coefficients checks equality exactly, using $(\sqrt{2})^2 = 2$, $(\sqrt{3})^2 = 3$ and $\sqrt{2}\sqrt{3} = \sqrt{6}$. For example, the first interior center in $\mathcal{Q}_3$ is $(\sqrt{6}, \sqrt{2})$ and its first boundary neighbor is $((\sqrt{2} + \sqrt{6})/2, (\sqrt{6} + 3\sqrt{2})/2)$. Their squared distance is

$$\left(\frac{\sqrt{2} - \sqrt{6}}{2}\right)^2 + \left(\frac{\sqrt{6} + \sqrt{2}}{2}\right)^2 = 4.$$

This exact arithmetic also identifies contacts between centers that are nonconsecutive in the defining chains.

Proposition 10.3 (Feasibility and contacts of the six references). *Each reference $\mathcal{Q}_1, \dots, \mathcal{Q}_6$ is feasible in radius b . Its exact numbers of centers, pair contacts and boundary contacts are those in Table 6. Every pair outside the contact list has distance strictly greater than 2, and every center outside the boundary-contact list has radius strictly less than b .*

Computation. Construct all coordinates from Table 5 and (10.5), adding O . Evaluate each squared norm and each squared pair distance in $\mathbb{Q}(\sqrt{2}, \sqrt{3})$. Identify pair contacts by exact equality to 4, and boundary contacts by exact equality of the squared norm to b_{18}^2 . For every other pair require a certified strict lower bound greater than 4, and require every other squared norm to be less than b_{18}^2 . Rational enclosures of the radicals certify these strict comparisons. The exact equalities give the contact counts in Table 6. $\square$

References	Centers	Pair contacts	Boundary contacts
$\mathcal{Q}_1, \mathcal{Q}_2$	18	34	10
$\mathcal{Q}_3, \mathcal{Q}_4, \mathcal{Q}_5$	18	32	11
$\mathcal{Q}_6$	17	31	10

Table 6: Exact numbers of centers and contacts in the six references.

The references are therefore feasible sets with fully specified contacts. Figure 11 shows their different radius types and the vacant boundary arc in $\mathcal{Q}_6$. The next step is to prove that every

unknown set with ten or eleven outer centers matches one of these references within a small radial error.

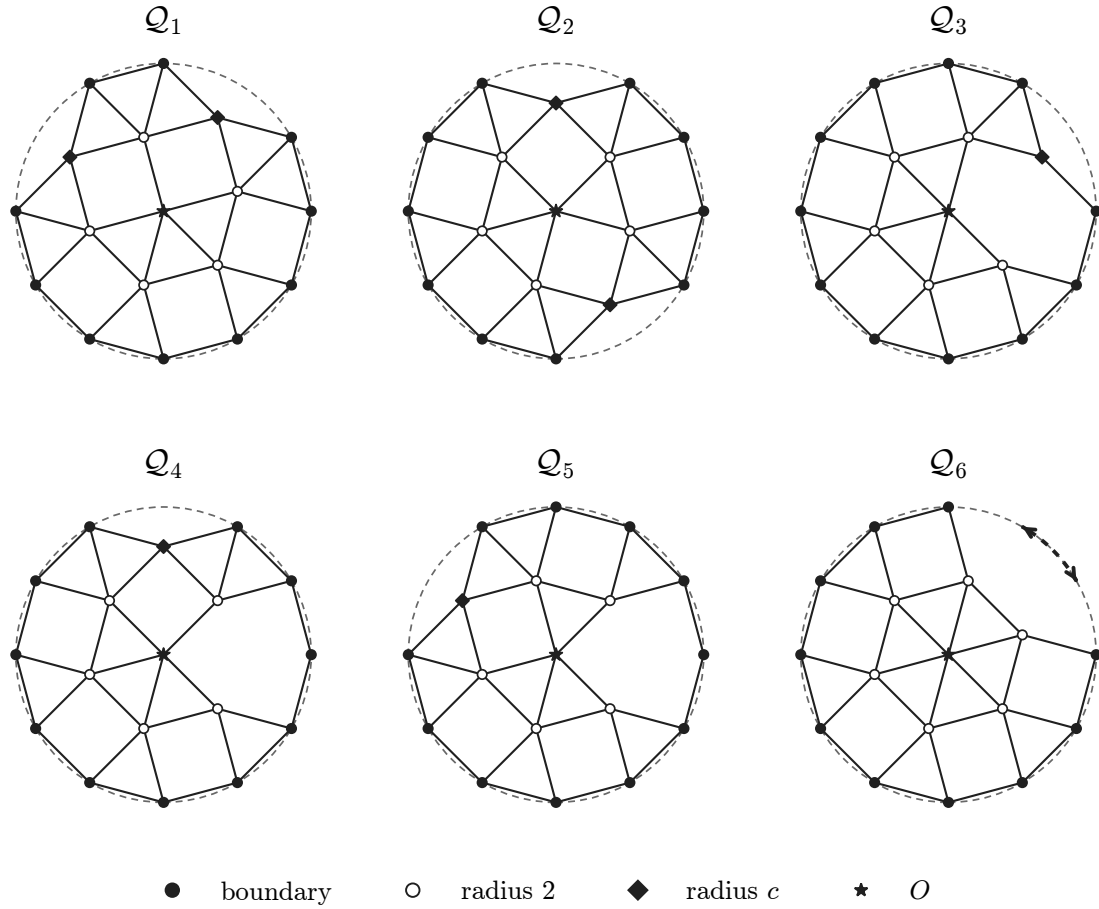


Figure 11: The six reference sets of Table 5. Filled circles mark boundary centers, open circles mark radius-2 centers, diamonds mark cutoff-radius centers, and stars mark O . Dashed circles have radius b_{18} . Segments join contact pairs. The marked arc in Q_6 runs from direction $\pi/6$ to $\pi/3$ and supports the additional center in the optimal family.

10.4 Matching the radial intervals and orders

Put

$$\varepsilon := 10^{-6}. \quad (10.7)$$

The matching test chooses one unknown inner center to correspond to O and requires its radius interval to lie in $[0, \varepsilon]$. It then compares the cyclic list of all other inner centers with one row of Table 5. Each radius must be within ε of its reference radius, and the intervening outer counts must match under the same cyclic shift or reversal. For Q_6 , use counts $(1, 2, 2, 2, 2, 2)$ in this comparison; the sole outer center in the first gap will later be set aside when matching the seventeen-point reference.

Deleting an entry from a cyclic list joins its two neighboring gaps, so their outer counts add. Table 7 illustrates this bookkeeping for a point matched to O . Its two adjacent counts are 1 and 1, which combine to 2. The remaining targets and counts are those of Q_3 . The chosen center remains part of the feasible set and is matched to O in the local argument.

Radius target	c	2	2	0	2	2	2
Following outer count	1	2	1	1	2	2	2

Table 7: An exact target list illustrating the origin match. The zero target requires a radius in $[0, \varepsilon]$; other targets allow error ε .

Proposition 10.4 (Matching to the six reference sets). *Every projected feasible eighteen-center set in radius b with ten or eleven outer centers has one of the radial and cyclic matches defined above, with tolerance $\varepsilon = 10^{-6}$. For ten outer centers the reference is $\mathcal{Q}_1$ or $\mathcal{Q}_2$; for eleven it is one of $\mathcal{Q}_3, \dots, \mathcal{Q}_6$. The match includes one center of radius at most ε corresponding to O and uses a common cyclic shift or reversal for all other radial targets and counts.*

Computation. Proof strategy. We enumerate every permitted cyclic order and cover its radial ranges by closed boxes. Valid inequalities exclude boxes; the remaining boxes must lie wholly in one of the stated matches. Complete resolution of these boxes gives the proposition by Proposition 4.2.

For $(m, k) = (11, 7)$ and $(10, 8)$, enumerate all nonnegative k -entry count vectors with sum m . The totals are 12 376 and 19 448 vectors, reduced by shifts and reversal to 912 and 1282 representatives. For every representative, start each radius in

$$[0, 2.828427124746191],$$

rounded outward to denominator 2^{40} . This contains $[0, c]$, as is checked by squaring the endpoint and comparing with 8. Apply radius contraction, chain-angle sums and directed angle closure, using 6340 certified angle bounds for $m = 11$ and 3290 for $m = 10$.

After every valid exclusion test, try each inner entry as the origin match and each common cyclic shift or reversal of the remaining radial targets and counts. A box is accepted only if the entire matched intervals satisfy the conditions just stated. For a target c , the acceptance interval is

$$[2.828426124746191, 2.828428124746190] \subset [c - \varepsilon, c + \varepsilon].$$

The inclusion follows by squaring the rational lower and upper bounds for $2\sqrt{2}$. For $m = 10$, permitted matches are $\mathcal{Q}_1, \mathcal{Q}_2$; for $m = 11$, they are $\mathcal{Q}_3, \dots, \mathcal{Q}_6$, with the modified last count vector stated above.

The $m = 11$ calculation examines 589 172 boxes and accepts 524; the $m = 10$ calculation examines 1 299 422 boxes and accepts 100. Every other terminal box is excluded. Both calculations finish with no unresolved case. The preceding geometric bounds place every projected feasible set in these starting ranges. The order enumeration covers every cyclic list by Section 4.2. Since every terminal box is excluded or wholly contained in one of the stated matches, Proposition 4.2 gives the conclusion for every feasible set. $\square$

For a match to the first five references, retain all eighteen centers. For a match to $\mathcal{Q}_6$, retain the origin match and the six other inner centers, together with the ten outer centers outside the first gap. These seventeen centers have the order of $\mathcal{Q}_6$ and remain pairwise separated. We will apply local rigidity to this retained subset.

10.5 The local-rigidity thresholds

Each reference contains $(0, -b_{18})$ as a boundary center. Align its matched outer center with that point. The following computation proves the positive stress identity (5.14) and the coordinate bound (5.15) after the horizontal displacement of that aligned center is fixed to zero.

Proposition 10.5 (Six local-rigidity certificates). *For each of the six references, fixing the horizontal coordinate of $(0, -b)$ gives the stress and inverse bounds (5.14) and (5.15), with the constants in Table 8.*

Computation. For each reference, use all pair and container contacts from Proposition 10.3. Form the signed rigidity matrix, delete the aligned horizontal coordinate, and adjoin the column of ones and the rational additional columns specified in Appendix G. Evaluate its residual against the supplied rational inverse proposal. The residual norm is less than 10^{-10} , and both inequalities (5.26) hold with the constants in Table 8. Exact interval evaluation verifies every condition for all six references. Lemma 5.6 and Proposition 5.5 convert these bounds into the positive stress identity (5.14) and the coordinate bound (5.15) for each reference. $\square$

References	n_{ref}	n_{con}	κ	σ_*	Threshold
$\mathcal{Q}_1, \mathcal{Q}_2$	18	44	3	1/64	1/1536
$\mathcal{Q}_3, \mathcal{Q}_4, \mathcal{Q}_5$	18	43	3	1/100	1/2400
$\mathcal{Q}_6$	17	41	4	1/70	1/2240

Table 8: Exact local-rigidity constants. The threshold is $\sigma_*/(8\kappa)$; n_{ref} and n_{con} count retained centers and selected constraints.

Proposition 10.5 gives a positive equilibrium stress and the inverse bound for each reference. It remains to place every matched center strictly inside the corresponding position threshold. We obtain this from the fact that consecutive noncentral reference centers are in contact and their gaps sum to one full turn.

10.6 From the radial match to a position bound

Write $\mathbf{x}_{\mathbf{p}}$ for the retained center matched to a reference center $\mathbf{p}$. Consider the cyclic list of all retained centers except the one matched to O . Every consecutive reference pair is a contact pair by the chain construction. Its angular gap is $\pi/12$, $\pi/6$ or $\pi/3$, according to the pair type. We bound how much such a gap can decrease when the endpoint radii have their allowed errors.

Proposition 10.6 (Derivatives of the contact angles). *The contact-angle functions are differentiable throughout the closed domains in Table 9 and satisfy all the strict derivative bounds in that table.*

Computation. For the exact radius b_{18} and $\varepsilon = 10^{-6}$, evaluate the derivative in (5.2) of Lemma 5.1. The domains and strict derivative bounds are given in Table 9. Rational interval bounds give positive radii, strict triangle inequalities and a positive lower bound on $\mathcal{D}$ throughout each domain. Lemma 5.1 therefore applies. Evaluating its derivative formulas with the same outward rounding gives the stated strict upper bounds. Appendix F gives the calculation. $\square$

Domain	Strict upper bounds
$r \in [2 - \varepsilon, 2 + \varepsilon]$	$ h'(r) < .867 < 1$
$r \in [c - \varepsilon, c + \varepsilon]$	$ h'(r) < .095 < 1$
$r, s \in [2 - \varepsilon, 2 + \varepsilon]$	$ g_r(r, s) , g_s(r, s) < .289 < 1/2$

Table 9: Exact rational bounds for the derivatives used in the gap comparison. The full intervals are included, also above the cutoff in the second row.

Lemma 10.7 (Position bounds from the contact gaps). *Suppose a feasible retained set has an accepted match to one of the six references, as obtained from Proposition 10.4. After alignment at $(0, -b_{18})$, every position error is strictly less than the bound in Table 10.*

Proof. Proof strategy. First we bound the loss in one gap by the radius errors and the certified derivatives. The gaps in each set sum to 2π , so a bound on possible losses also bounds the change of any consecutive block of gaps. Such a block measures a direction relative to the aligned point. The polar position estimate then gives the result.

Let δ_j^* and δ_j be corresponding reference and unknown consecutive gaps. Their sums satisfy

$$\sum_j \delta_j = \sum_j \delta_j^* = 2\pi. \quad (10.8)$$

For an outer–outer pair the radii remain b_{18} , so the unknown gap is at least the reference contact angle $\pi/6$. For a mixed pair, let r^* be its reference interior radius, either 2 or c . The mean value theorem and Proposition 10.6 give

$$|h(r) - h(r^*)| \leq \varepsilon.$$

The unknown gap is at least the first of these contact angles, whereas the reference gap equals the second. Its loss is therefore at most ε .

The only consecutive interior–interior pair occurs in $\mathcal{Q}_6$, with both reference radii 2. Change its two radii one at a time and apply the last derivative bound:

$$\begin{aligned} |g(r, s) - g(2, 2)| &\leq |g(r, s) - g(2, s)| + |g(2, s) - g(2, 2)| \\ &\leq \frac{1}{2}|r - 2| + \frac{1}{2}|s - 2| \leq \varepsilon. \end{aligned}$$

Combining the pair types proves

$$\delta_j - \delta_j^* \geq \begin{cases} 0, & \text{both reference endpoints are outer,} \\ -\varepsilon, & \text{otherwise.} \end{cases} \quad (10.9)$$

Let n_{loss} be the number of gaps in the second case. For $\mathcal{Q}_1, \mathcal{Q}_2$, seven noncentral interior centers each have two outer neighbors, giving $n_{\text{loss}} = 14$. For $\mathcal{Q}_3, \mathcal{Q}_4, \mathcal{Q}_5$, six such centers give $n_{\text{loss}} = 12$. For $\mathcal{Q}_6$, counting two sides of each of its six interior centers counts the one interior–interior gap twice; there are $12 - 1 = 11$ distinct gaps that can lose angle.

For any consecutive block $\mathcal{J}$ of gaps, define its total change by

$$\Delta_{\mathcal{J}} := \sum_{j \in \mathcal{J}} (\delta_j - \delta_j^*). \quad (10.10)$$

Equation (10.9) gives $\Delta_{\mathcal{J}} \geq -n_{\text{loss}}\varepsilon$, and the same bound holds for its complementary block. Their changes sum to zero by (10.8). Hence

$$-n_{\text{loss}}\varepsilon \leq \Delta_{\mathcal{J}} = -\Delta_{\mathcal{J}^c} \leq n_{\text{loss}}\varepsilon.$$

After alignment, each noncentral angular error is the change of such a block and is therefore at most $n_{\text{loss}}\varepsilon$. Its radial error is at most ε by the accepted match. Lemma 5.3 gives

$$\|\mathbf{x}_{\mathbf{p}} - \mathbf{p}\| \leq (1 + b_{18}n_{\text{loss}})\varepsilon.$$

The center matched to O has error at most ε directly from its radius. Finally, use $b_{18} < 3.864$ and $\varepsilon = 10^{-6}$ to obtain the strict bounds in Table 10, including the origin match. $\square$

References	n_{loss}	$1 + 3.864n_{\text{loss}}$	Strict error bound
$\mathcal{Q}_1, \mathcal{Q}_2$	14	55.096	$56 \cdot 10^{-6}$
$\mathcal{Q}_3, \mathcal{Q}_4, \mathcal{Q}_5$	12	47.368	$48 \cdot 10^{-6}$
$\mathcal{Q}_6$	11	43.504	$44 \cdot 10^{-6}$

Table 10: Position bounds from the gap comparison. All table entries are exact; the last column gives strict rational upper bounds.

10.7 The lower bound

The position errors in Lemma 10.7 satisfy

$$56 \cdot 10^{-6} < 1/1536, \quad 48 \cdot 10^{-6} < 1/2400, \quad 44 \cdot 10^{-6} < 1/2240.$$

Each Euclidean bound also bounds every coordinate error. The aligned point has zero horizontal displacement, and Proposition 10.5 supplies the positive stress and inverse bound through the residual lemma. All selected distance and containment inequalities hold for the retained feasible set. Thus every hypothesis of Lemma 5.4 holds in each match, and each retained center equals its reference position.

Each of $\mathcal{Q}_1, \dots, \mathcal{Q}_5$ has a center at radius exactly c matched to an unknown inner center. Exact agreement contradicts that center's strict projected bound $r < c$. Therefore $m = 10$ is excluded, and only the $\mathcal{Q}_6$ match remains when $m = 11$. In that match, the boundary center of direction 0 touches the radius-2 center of direction $\pi/12$. Since $2 < c$, Lemma 2.3 implies that this outer center originally had radius b_{18} .

The $m = 12$ case already supplied an original boundary center. All cases in (10.4) have now either been excluded or shown to require one. Thus no smaller center disk is possible. Together with (10.3), this proves

$$R_{18} = 1 + \sqrt{2} + \sqrt{6}. \quad (10.11)$$

10.8 A continuous family of optimal packings

The seventeen-center reference $\mathcal{Q}_6$ can be augmented by $b_{18}\mathbf{e}(\theta)$ for any $\theta \in [\pi/6, \pi/3]$. Its nearest boundary neighbors have directions 0 and $\pi/2$, so both angular separations are at least $\pi/6$. Its nearest radius-2 neighbors have directions $\pi/12$ and $5\pi/12$, so both separations are at least $\pi/12$. The chord formula and (10.2) give distance at least 2 from these four neighbors. All other noncentral reference centers have larger smaller angular separations of the same pair types. Its distance from O is $b_{18} > 2$. Thus the added center gives a feasible eighteen-center set for every parameter in the stated interval.

These sets are optimal by (10.11) and are precisely the family displayed in (1.2). At $\theta = \pi/6$, the largest gap between boundary directions is $\pi/3$; at $\theta = \pi/4$, it is $\pi/4$. An isometry carrying both the packing and its container to another preserves these gaps, so those two members are not congruent. This proves the asserted nonuniqueness. The lower-bound argument for $m = 12$ does not classify its six inner centers, so the family assertion has exactly the scope stated in Theorem 1.1.

Figure 12 displays the two members used to show noncongruence. The marked center moves on radius b through the angular range indicated by the arrows outside the container; all seventeen other centers stay fixed.

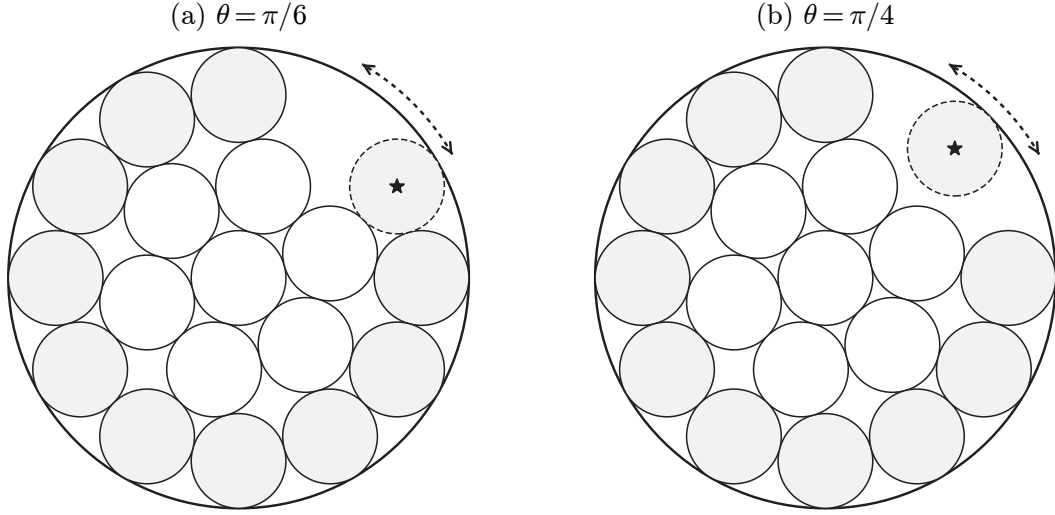


Figure 12: Two members of the optimal eighteen-circle family. A star marks the variable center, its unit circle has a dashed outline, and the arrowed arc outside the container indicates the allowed angular range $[\pi/6, \pi/3]$. The parameter values are $\pi/6$ and $\pi/4$.

11 Twenty circles

Proof strategy. The construction contains a center that can move, so the proof will determine a fixed subset of nineteen centers. First the count and radius bounds reduce an arbitrary projected set to twelve outer and eight inner centers. The finite computation selects seven of the inner centers and bounds their radii and order. After these nineteen centers are matched to the reference, position bounds and local rigidity force exact agreement. Recovering their original positions proves the lower bound and gives the full condition on the remaining center.

11.1 A fixed nineteen-center set and a free center

The construction has twelve outer centers and seven interior centers fixed by contacts. A closing contact at the top determines the radius. The twentieth center can then be placed near O , with room to move. We first specify the nineteen fixed centers, so that the final equality statement can describe every permissible position of the remaining center.

Let b vary over the exact rational interval

$$\begin{aligned} b_{20}^- &= 4.1223207369915283214476857921, \\ b_{20}^+ &= 4.1223207369915283214476857923. \end{aligned} \tag{11.1}$$

Start at $\mathbf{A} = (1, -\tau_b)$, with $\tau_b = \sqrt{b^2 - 1}$, and use the boundary rotation (6.1) to define

$$\begin{aligned} \mathbf{C} &= R_b \mathbf{A}, & \mathbf{H} &= R_b^2 \mathbf{A}, & \mathbf{M} &= R_b^3 \mathbf{A}, \\ \mathbf{U} &= R_b^4 \mathbf{A}, & \mathbf{L} &= R_b^5 \mathbf{A}. \end{aligned}$$

These six centers and their reflections in the vertical axis have radius b . They form a chain of eleven contacts: five on each side and the bottom pair $\mathbf{A}, \mathbf{SA}$.

Place the first inner center above $\mathbf{A}$. Use the common distance-2 intersections (6.2) to define the next two:

$$\mathbf{F} := \mathbf{A} + (0, 2), \quad \mathbf{E} := \mathbf{l}_+(\mathbf{F}, \mathbf{H}), \quad \mathbf{J} := \mathbf{l}_+(\mathbf{E}, \mathbf{U}).$$

These give the five contacts **AF**, **FE**, **HE**, **EJ** and **UJ**. The horizontal coordinate of **F** is 1, so it also touches its reflection. To complete the inner set, place an axial center above **J** at distance 2:

$$\mathbf{K} := \left(0, J_y + \sqrt{4 - (J_x)^2}\right), \quad (11.2)$$

where (J_x, J_y) are the coordinates of **J**. It also touches **SJ**. The final contact requires it to touch **L**, giving

$$F_{20}(b) := \|\mathbf{L} - \mathbf{K}\|^2 - 4 = 0. \quad (11.3)$$

Reflection gives the other closing contact with **SL**.

The proposed nineteen-center set consists of the twelve outer centers, the three centers **F**, **E**, **J** and their reflections, and the axial center **K**. Figure 13 shows this set with the twentieth center placed at *O*.

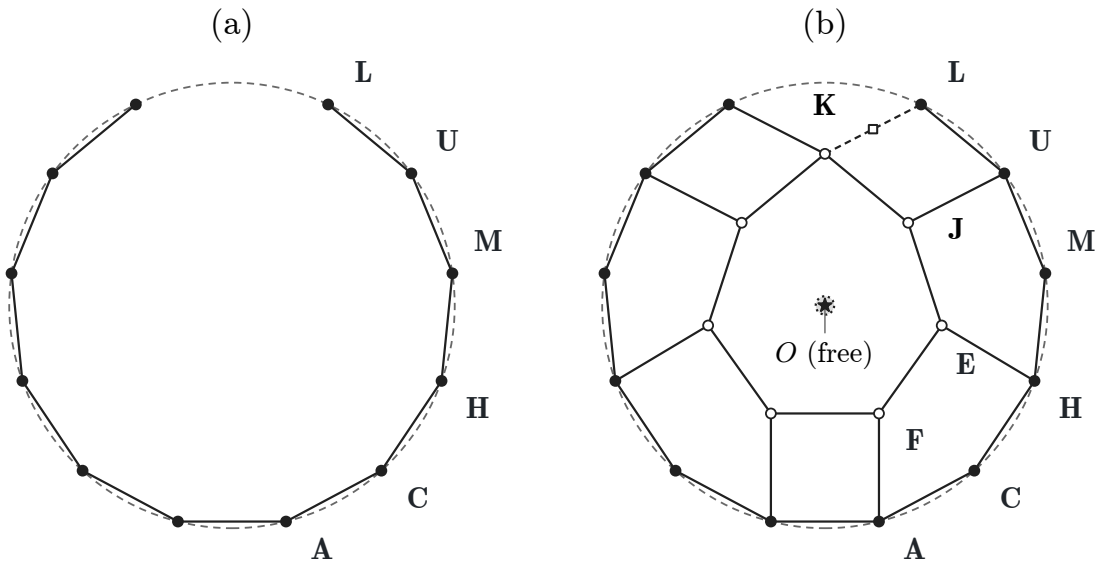


Figure 13: The twenty-center construction. The outer chain runs from **SL** through **SA**, **A** to **L**. Filled and open circles are outer and inner centers; a star marks the free center at *O*. The dashed outer circle has radius b . Segments are distance-2 contacts. The marked dashed contact between **L** and **K** determines the radius. The small dotted circle about *O* has radius $17/100$ and lies in the feasible region for the free center.

Proposition 11.1 (The twenty-center construction). *Throughout $[b_{20}^-, b_{20}^+]$, the coordinate formulas have nonzero denominators and strictly positive radicands. The endpoint values of F_{20} have opposite signs. At every root in this interval, the stated construction is a feasible set of 20 centers, with all its proposed inner centers strictly inside $c(b)$. For every pair outside the selected contact list, $\|\mathbf{p} - \mathbf{q}\|^2 - 4 > .2346221317$. Each proposed inner center also satisfies $b^2 - \|\mathbf{p}\|^2 > 9.1237296549$. The derivative satisfies $F'_{20}(b) > 16.035$, and every noncentral center has radius greater than 2.17. Hence the closing equation has exactly one root in the interval.*

Computation. Proof strategy. We first verify that the coordinate formulas are defined throughout the input interval and that the closing function changes sign. We then bound all noncontact distances and inner radii. The exact contact identities and the intermediate value theorem give the feasible construction.

Evaluate the coordinate formulas over $[b_{20}^-, b_{20}^+]$, including a twentieth center at *O*. Certify every denominator to be nonzero and every radicand to be strictly positive. The endpoint

values of F_{20} have opposite signs, and its derivative satisfies $F'_{20}(b) > 16.035$ throughout the interval. Table 13 prints rational enclosures for these values and the domain margins.

The selected 26 pair contacts are the eleven outer-chain contacts, the three remaining edges of the bottom square, the eight contacts from the two circle intersections and their reflections, and the four contacts of $\mathbf{K}$ with $\mathbf{J}, \mathbf{SJ}, \mathbf{L}, \mathbf{SL}$. For each of the other 164 pairs, certify

$$\|\mathbf{p} - \mathbf{q}\|^2 - 4 > .2346221317.$$

For each of the eight proposed inner centers, certify $b^2 - \|\mathbf{p}\|^2 > 9.1237296549$ and $\|\mathbf{p}\| < c(b)$. For every noncentral center, certify $\|\mathbf{p}\| > 2.17$. Rational interval evaluation gives these strict bounds. The positive domain margins make F_{20} continuous, so the opposite endpoint signs give a root by the intermediate value theorem. At a root, the closing contact is exact. Every other selected contact is exact by the preceding construction identities, and the remaining distance and containment bounds establish feasibility. The strict derivative sign makes the root unique. $\square$

The strict domain tests make F_{20} continuously differentiable on the interval. Opposite endpoint signs give a root, and the positive derivative makes it unique. Denote it by b_{20} , and let $\mathcal{C}$ be the nineteen noncentral centers evaluated there. Let $\mathcal{P}_{20} := \mathcal{C} \cup \{O\}$. All selected pair distances equal 2 by the construction formulas, except the final two, which follow from $F_{20}(b_{20}) = 0$. Proposition 11.1 proves all remaining pair inequalities and the inner containment; the twelve outer radii are exactly b_{20} . Thus

$$R_{20} \leq 1 + b_{20}. \quad (11.4)$$

To describe the available positions of the last center, use the region defined in Theorem 1.1:

$$\mathcal{A} = \{\mathbf{z} : \|\mathbf{z}\| \leq b_{20}, \|\mathbf{z} - \mathbf{p}\| \geq 2 \text{ for every } \mathbf{p} \in \mathcal{C}\}. \quad (11.5)$$

Every point with $\|\mathbf{z}\| \leq .17$ belongs to this region, because every $\mathbf{p} \in \mathcal{C}$ has norm greater than 2.17:

$$\|\mathbf{p} - \mathbf{z}\| \geq \|\mathbf{p}\| - \|\mathbf{z}\| > 2.17 - .17 = 2. \quad (11.6)$$

Also $\|\mathbf{z}\| < b_{20}$. Hence the center at O is free to move throughout this disk. A packing disk whose center can move while all other centers stay fixed is called a *rattler*. Here its full feasible region is specified by (11.5).

11.2 The numbers and radii of inner centers

Set $b = b_{20}$ and project an arbitrary feasible twenty-center set. We write $c = c(b)$, $\eta = \eta(b)$, $h(r) = h_b(r)$ and $f(r) = f_b(r)$ for this fixed radius. We first certify the common angular and count hypotheses, together with the radius cap needed for the eventual order calculation.

Proposition 11.2 (Geometric constants for twenty centers). *Take*

$$b_- = 4.122320736, \quad b_+ = 4.122320737, \quad c_+ = 3.151994, \quad r_*^- = 2.26714.$$

Then all hypotheses of Lemma 2.6 and (4.7) hold, and

$$13\eta > \pi, \quad 10 \arcsin(1/3.151994) > \pi, \quad 3.151994 - 4/3.151994 < 2.$$

For the case $m = 12$, put

$$\omega_{12}^+ := 3.141592653589793239 - 11(.245026263334771).$$

We have $\eta > .245026263334771$ and

$$0 < \omega_{12} < \omega_{12}^+ < \pi/2, \quad c < b_{20}^- \cos \omega_{12}^+, \quad D(2.806, b_{20}^+, \omega_{12}^+) < 4. \quad (11.7)$$

Computation. We evaluate the formulas of Sections 2–3 on the whole defining interval $[b_{20}^-, b_{20}^+]$, using the outward-rounded rational arithmetic and trigonometric bounds of Appendix B. For each strict comparison, the upper endpoint on its smaller side is below the lower endpoint on its larger side. The four comparisons (2.14) are evaluated separately, as are the domain conditions and each displayed bound. These finite comparisons give all the stated inequalities; the same enclosures verify every hypothesis named in the proposition. $\square$

The outer angle bound gives $m \leq 12$. Apply Lemma 2.8 with $c_+ = 3.151994$, $\nu = 10$ and $\mu = 6$. Its small-disk maximum equals $d^2 < 4$, because $2(1 - \cos(\pi/3)) = 1$ and $d = c_+ - 4/c_+ < 2$. Thus $k \leq 14$, and $m + k = 20$ leaves $8 \leq k \leq 14$.

Proposition 11.3 (Excluding nine to fourteen inner centers). *Let $k \in \{9, 10, 11, 12, 13, 14\}$ and let $\mathbf{x}_1, \dots, \mathbf{x}_k$ be centers with pairwise distances at least 2 and radii $r_i = \|\mathbf{x}_i\| \in [0, 3.151994]$. Then*

$$\sum_{i=1}^k f(r_i) > \pi - (20 - k)\eta.$$

Computation. We exclude the opposite weak inequality on all possible radius vectors. For $k = 9, \dots, 14$, begin with $[0, 3.151994]^k$, rounded outward on the radial grid of denominator 2^{30} , in inner cyclic order. Use the labeling inequalities (4.2), radius contraction, certified four-corner angle bounds, directed angle closure, and (4.10) with $m = 20 - k$. The completed calculation excludes all such boxes for all six counts, with no unresolved box. Appendix C records their exact numbers. The initial boxes contain every radius vector in the statement, and Section 4.2 covers every cyclic order, including ties. Proposition 4.2 therefore excludes the weak inequality for every such separated set, proving the strict inequality. $\square$

The projected inner set lies in the starting ranges. Proposition 11.3 then contradicts the angular-length bound in Lemma 2.6, whose hypotheses were checked in Proposition 11.2, whenever $9 \leq k \leq 14$. Therefore

$$k = 8, \quad m = 12.$$

Now apply Lemma 3.2 with $r_{\text{cap}} = 2.806$, the defining endpoints $b_{20}^\pm$, and the bounds (11.7). It gives

$$0 \leq r_i < 2.806 \quad (1 \leq i \leq 8). \quad (11.8)$$

The largest reference inner radius, that of $\mathbf{K}$, is near 2.805316133, so this cap includes the construction.

11.3 Selecting seven inner centers

The eighth inner center need not remain close to O . We therefore ask the finite calculation to identify seven inner centers whose radii and intervening outer counts match the reference. All twenty centers participate in the separation tests. The target comparison then tries each of the eight choices of an inner center to set aside, merging its two adjacent outer counts.

For example, if the selected entry has two outer centers before it and one after it, the count between its former inner neighbors becomes $2 + 1 = 3$. In general the merged count is $q_{i-1} + q_i$. The remaining counts and radius targets are compared together under a common cyclic shift or reversal, using the order rule of Section 4.2.

Proposition 11.4 (A nineteen-center match). *Every projected feasible twenty-center set in radius b contains a matched subset of all twelve outer centers and seven inner centers. After a cyclic shift or reflection, these seven inner centers correspond to*

$$(\mathbf{J}, \mathbf{K}, \mathbf{S}\mathbf{J}, \mathbf{S}\mathbf{E}, \mathbf{S}\mathbf{F}, \mathbf{F}, \mathbf{E})$$

Reference centers	Exact rational radius target
J, SJ	2.171529956399
K	2.805316132579
E, SE	2.193139842425
F, SF	2.235344370918

Table 11: Targets for the seven matched inner radii. Each acceptance interval has radius 10^{-5} about its listed exact target.

with following outer counts $(1, 1, 2, 2, 2, 2, 2)$. Each matched radius differs by at most 10^{-5} from the corresponding rational target in Table 11.

Computation. Proof strategy. We enumerate every permitted cyclic order and cover its radial ranges by closed boxes. Valid inequalities exclude boxes; the remaining boxes must lie wholly in one of the stated matches. Complete resolution of these boxes gives the proposition by Proposition 4.2.

Enumerate all nonnegative eight-entry count vectors with sum 12. There are $\binom{19}{7} = 50\,388$ vectors and 3260 dihedral representatives. Start all eight radii in $[0, 2.806]$, rounded outward to denominator 2^{40} , and apply radius contraction, chain-angle sums and directed angle closure to all twenty centers, using 26 878 certified pair-angle bounds.

For each remaining box, try every choice of one inner entry to omit from the target comparison. We retain the whole box only when one such choice satisfies all the stated order and radial-target conditions. The completed calculation examines 5 822 866 boxes and accepts 5406 under these conditions. Every other terminal box is excluded, and no unresolved case remains. The preceding geometric bounds place every projected feasible set in these starting ranges. The order enumeration covers every cyclic list by Section 4.2. Since every terminal box is excluded or wholly contained in one of the stated matches, Proposition 4.2 gives the conclusion for every feasible set. $\square$

Use the match supplied by Proposition 11.4. Retain those seven inner centers and all twelve outer centers. They form a feasible nineteen-center set. For $\mathbf{p} \in \mathcal{C}$, write $\mathbf{x}_{\mathbf{p}}$ for its matched unknown center in the resulting full reference order

$$(\mathbf{J}, \mathbf{L}, \mathbf{K}, \mathbf{SL}, \mathbf{SJ}, \mathbf{SU}, \mathbf{SM}, \mathbf{SE}, \mathbf{SH}, \mathbf{SC}, \mathbf{SF}, \mathbf{SA}, \mathbf{A}, \mathbf{F}, \mathbf{C}, \mathbf{H}, \mathbf{E}, \mathbf{M}, \mathbf{U}). \quad (11.9)$$

The next steps bound the positions of these nineteen centers and then force agreement with $\mathcal{C}$.

11.4 Position bounds and local rigidity

Align the outer center matched to **SA** with that reference position. Their common radius b_{20} makes the two positions agree. We will certify the common order and radial ranges, use the directed angle widths to bound displacement, and compare that bound with the local-rigidity threshold.

Proposition 11.5 (Position bounds for the nineteen matched centers). *Let $\{\mathbf{x}_{\mathbf{p}} : \mathbf{p} \in \mathcal{C}\}$ be a feasible matched set in radius b with the full cyclic order (11.9) and the closed radial intervals of Table 11. Assume every matched outer center has radius b . Align the center matched to **SA** with that reference point. Then*

$$\|\mathbf{x}_{\mathbf{p}} - \mathbf{p}\| < .00011675 \quad (\mathbf{p} \in \mathcal{C}).$$

Computation. Proof strategy. We first place the reference in the same radial ranges and cyclic order as the unknown set. Common directed angle bounds then control the differences of their directions. The polar position estimate converts these angular and radial bounds to the asserted distances.

Use the seven closed target intervals of Table 11, the order (11.9), and $[b_{20}^-, b_{20}^+]$. Interval norms place the reference radii in the targets. Coordinate signs and pair determinants certify its cyclic order, using the signs to select half turns before applying the determinant comparison.

Certify all four-corner pair-angle bounds and close the directed inequalities (4.4), with $\Pi^+ = 6908435304716/2^{40}$. Relative to the aligned center, the widths are those in Table 12. Every resulting width is nonnegative. Substitution in (5.8) gives a bound strictly below .00011675 for each center. Both the unknown and reference radii lie in the same intervals, so the radial errors are at most $2 \cdot 10^{-5}$ for inner centers and zero for outer centers. The directed bounds enclose both sets' relative directions, so their full widths bound the angular errors. Lemma 5.3 gives the claimed distances. $\square$

Reference centers	$2^{40}\varepsilon_{\theta,i}$	Strict upper bound for $\varepsilon_{\text{pos},i}$
F, SF	25 805 253	$116.750 \cdot 10^{-6}$
E, SE	23 929 317	$109.717 \cdot 10^{-6}$
J, SJ	21 680 556	$101.286 \cdot 10^{-6}$
K	4 023 210	$35.084 \cdot 10^{-6}$
Outer centers other than SA	4 023 210	$15.084 \cdot 10^{-6}$

Table 12: Exact angle-width numerators and rational strict upper bounds for position errors. The aligned center has zero displacement.

Appendix D.3 gives the chain additions for the largest bound, that of **F**.

Proposition 11.6 (Local-rigidity certificate for nineteen centers). *For the selected pair and container contacts, fix the horizontal coordinate of **SA**. The signed linear constraints satisfy (5.14) and (5.15) with $\sigma_* = 1/150$ and $\kappa = 7$. Thus the local-rigidity threshold of Lemma 5.4 is $\sigma_*/(8\kappa)$.*

Computation. Use the 26 pair contacts of $\mathcal{C}$ and its twelve boundary contacts. Delete the horizontal coordinate of **SA**, giving 37 variables and 38 signed constraints. Adjoin a column of ones. Against the supplied rational inverse proposal, certify a residual norm less than 10^{-10} and both inequalities (5.26) with

$$\kappa = 7, \quad \sigma_* = 1/150, \quad \frac{\sigma_*}{8\kappa} = 1/8400.$$

These inequalities hold throughout the defining interval. The contact indexing and rational matrix are specified in Appendix G. Lemma 5.6 proves invertibility and the stated inverse bound from this residual. Proposition 5.5 then supplies a normalized stress with every weight greater than σ_* and the displacement bound with constant κ . This proves the positive stress identity (5.14) and the coordinate bound (5.15). $\square$

The position bound is strictly below the threshold, with the exact margin

$$\frac{1}{8400} - .00011675 = \frac{193}{84000000} > 0. \quad (11.10)$$

Thus every coordinate displacement satisfies the required strict inequality. The aligned coordinate is fixed, and the matrix certificate gives the positive stress and inverse bound through the residual lemma. All selected separation and containment inequalities hold for the retained feasible set. Lemma 5.4 therefore gives $\mathbf{x}_{\mathbf{p}} = \mathbf{p}$ for every $\mathbf{p} \in \mathcal{C}$: the nineteen projected centers agree exactly with the reference.

11.5 The lower bound and all equality cases

The center **SA** touches the inner center **SF**. Lemma 2.3 puts its original radius at b_{20} . Therefore no smaller disk can contain the original twenty centers, irrespective of the remaining center's position. With (11.4), this proves

$$R_{20} = 1 + b_{20}. \quad (11.11)$$

We next recover all nineteen matched centers before projection, so that the same fixed subset occurs in every optimal packing.

The outer centers **A, H, U, L** touch, respectively, the unchanged inner centers **F, E, J, K**. Lemma 2.3 fixes their original radii and those of their reflected partners at b_{20} . The remaining outer centers are **C, M** and their reflections. Each touches one of the fixed outer centers, so Lemma 2.4 restricts its original radius to c or b_{20} .

Proposition 11.7 (Excluding cutoff positions for twenty centers). *For every $b \in [b_{20}^-, b_{20}^+]$, the construction coordinates satisfy*

$$\|(c(b)/b)\mathbf{C} - \mathbf{F}\|^2 < 1.3556, \quad \|(c(b)/b)\mathbf{M} - \mathbf{E}\|^2 < 1.5971.$$

The same bounds hold for the reflected pairs.

Computation. Evaluate the coordinate formulas on the whole closed interval $[b_{20}^-, b_{20}^+]$, using the certified domain margins in Table 13. Form each displayed difference, square its two coordinates and add. Outward-rounded interval evaluation gives an upper endpoint strictly below the stated rational bound. Reflection preserves squared distance, proving the remaining cases. $\square$

Both bounds are less than 4, so the cutoff positions violate separation from the unchanged inner centers. Every retained outer center originally had radius b_{20} . The original packing therefore contains an orthogonal image of $\mathcal{C}$ after its container center is moved to O .

Apply the inverse orthogonal transformation, and denote the remaining center by $\mathbf{z}$. Feasibility requires exactly $\|\mathbf{z}\| \leq b_{20}$ and $\|\mathbf{z} - \mathbf{p}\| \geq 2$ for every $\mathbf{p} \in \mathcal{C}$, so $\mathbf{z} \in \mathcal{A}$. Conversely, each point of $\mathcal{A}$ gives a feasible twenty-center set together with $\mathcal{C}$, by its definition (11.5). Every such set is optimal by (11.11). This proves the complete equality statement in Theorem 1.1, including the disk of free positions certified in (11.6).

12 Comparison with earlier proofs

The proofs above combine three steps: restrict radii and directions, exhaust all remaining interval cases, and determine the positions in each retained case. We compare these steps with earlier geometric proofs and with the rigidity principle used at the end. The constructions themselves have the history given in Section 1.

12.1 Radial restrictions and angular exclusions

Bateman and Erdős bound angular separation in an annulus and add widths of excluded sectors [1, Section 3, Lemma 1 and pp. 309–310]. Their annulus threshold is $s - 1/s$ when the minimum separation is 1. Doubling lengths and setting $b = 2s$ gives $b - 4/b = c(b)$. Lemma 2.2 uses this same threshold for radial projection. Lemma 2.3 then shows how a contact in the projected set can force the original center to have been on the boundary. The angular length inequality (2.15) also uses excluded directions: disjoint intervals of half-width $f_b(r)$ give additive restrictions on the outer count.

Fodor's thirteen-circle proof first separates points into an inner disk and an annulus. It reduces to three or four inner points, then uses angle sums, radial case divisions and diameter

bounds to identify the equality cases [12, Theorem 1 and Sections 1–2]. In our units its radius is $b = 1 + \sqrt{5}$ and the dividing radius is $c(b) = 2$. In particular, the diameter argument in his Lemma 5 and our Lemma 3.4 use the same elementary obstruction: a region of diameter less than the required separation cannot contain two centers. Fodor describes his method as a development of Bateman and Erdős’s method, also used for twelve and nineteen circles [12, Abstract].

Our finite subdivisions replace repeated hand divisions of radial ranges. Section 4 permits every radial choice in each box and every cyclic order covered by its initial cases. The directed angle bounds use nonconsecutive pairs as well as neighbors. The computation must either exclude a whole box or place it wholly in a specified final range. These final ranges still permit motion, which is why the position and local-rigidity arguments are separate steps.

12.2 Contact chains and local rigidity

Ekanayake and LaFountain introduce tight partitions, consisting of closed contact chains that make one turn around the origin. They obtain such chains after an outward-maximization step, with a qualification concerning outer rattlers [13, Definitions 2.1–2.2 and Theorem 2.1]. For fourteen circles, they reduce to ten boundary centers and four inner centers in a rhombus, then optimize the resulting one-parameter family [13, Theorems 3.1 and 4.1]. Their radial thresholds in Section 3.1 are our $c(b)$ and $r_*(b)$: the first solves $g(r, b) = 2\eta(b)$ on the inner branch, and the second solves $g(r, b) = \eta(b)$. Our global exclusions precede the choice of a contact pattern. Contacts are used after all centers have been placed near one of the stated reference sets.

The final argument is quantitative first-order rigidity. Positive weights cancel the linearized constraints, and full rank controls the coordinate changes by those constraints. The relation between first-order rigidity and rigidity is established in the literature; see [14, Section 2.2]. Our Lemma 5.4 gives the explicit estimate needed here: if the stress weights exceed σ_* and the inverse bound is κ , feasibility gives $U \leq 8\kappa U^2/\sigma_*$. Thus $U < \sigma_*/(8\kappa)$ forces exact agreement. The contribution of the certificate calculation is to prove constants whose neighborhoods contain every retained box. The fifteen-circle proof instead uses the odd cycle of five inner radii.

12.3 Scope of the computations and equality results

All five lower bounds depend on the exhaustive computations. The arithmetic validators certify every numerical premise, and the coverage proof explains why the finite subdivisions include every feasible set.

For eighteen circles, the proof establishes the least radius and the family in (1.2). It leaves the full equality classification open. For twenty circles, it establishes the exact description $\mathcal{C} \cup \{\mathbf{z}\}$ with $\mathbf{z} \in \mathcal{A}$, up to an orthogonal transformation. The proofs therefore give different kinds of equality information, as stated in Theorem 1.1. No computational exclusion for another number of circles is claimed here.

A Construction intervals and common geometric data

The case proofs use exact construction radii, together with coarser rational intervals for the shared geometric tests. We collect those intervals and the resulting count bounds here. The radical definitions of b_{15} and b_{18} , and the root intervals for b_{17} and b_{20} , are given in their construction sections; the longer interval for b_{16} is specified first.

A.1 The interval defining the sixteen-circle radius

The endpoints used in (8.5) are specified by the following rational numbers:

$$\begin{aligned} b_{16}^{\text{base}} &= 3.615425594873193985539244161991965 \\ &\quad + 247608277922833689762319592317 \cdot 10^{-63}, \\ b_{16}^- &= b_{16}^{\text{base}} + 63302032464297878 \cdot 10^{-80}, \\ b_{16}^+ &= b_{16}^{\text{base}} + 78348360154823159 \cdot 10^{-80}. \end{aligned}$$

All construction and coefficient bounds for sixteen circles hold throughout this interval.

A.2 Signed root and domain certificates

The construction computations certify more than the signs needed by the intermediate value theorem. Table 13 records outward-rounded rational enclosures for the endpoint values. It also records positive lower bounds for every square-root radicand and every absolute interval divisor evaluated in the coordinate formulas, throughout the closed defining interval. Rational coefficients are evaluated exactly.

For sixteen circles, the radicands are $b^2 - 1$ from the boundary rotation, $4 - (X_1 - 1)^2$ and $4 - (X_5)^2$ from (8.2)–(8.3), and $4/\|\mathbf{Z} - \mathbf{A}\|^2 - 1/4$ from (8.4). The interval divisors are b^2 , $\|\mathbf{Z} - \mathbf{A}\|^2$ and 2. Every listed radicand and divisor satisfies the corresponding lower bound in the $N = 16$ row.

For seventeen circles, the radicands are $b^2 - 1$, 3, $b^2/e - \alpha_E^2$ from (9.2), $4/\|\mathbf{M} - \mathbf{G}\|^2 - 1/4$ from (9.3), and $4 - (J_x)^2$ from (9.4). The interval divisors are b^2 , e , $2e$, $\|\mathbf{M} - \mathbf{G}\|^2$ and 2. The $N = 17$ row bounds every expression in these two lists.

For twenty circles, the radicands are $b^2 - 1$, $4/\|\mathbf{H} - \mathbf{F}\|^2 - 1/4$, $4/\|\mathbf{U} - \mathbf{E}\|^2 - 1/4$, and $4 - (J_x)^2$. They occur in the rotation, the two intersections (6.2), and (11.2). The interval divisors are b^2 , $\|\mathbf{H} - \mathbf{F}\|^2$ and $\|\mathbf{U} - \mathbf{E}\|^2$. The $N = 20$ row bounds every expression in these lists.

N	Scaled $F_N(b_N^-)$	Scaled $F_N(b_N^+)$	Radicand $\geq$	Divisor $\geq$
16	$[-1.183, -1.182]$	$[\text{.840}, \text{.841}]$	.127	2.000
17	$[3.671, 3.672]$	$[-4.002, -4.001]$	.033	2.000
20	$[-1.618, -1.617]$	$[1.590, 1.591]$	.181	8.648

Table 13: Exact rational enclosures and lower domain bounds. The endpoint values are multiplied by 10^{63} for $N = 16$ and by 10^{27} for $N = 17, 20$. Each interval endpoint is rounded outward; the domain margins are rounded downward.

The same interval differentiation gives $F'_{17}(b) \in [-38.367, -38.366]$ and $F'_{20}(b) \in [16.035, 16.036]$ throughout their defining intervals. The twenty-circle computation also checks the strict inequality $F'_{20}(b) > 16.035$ used in its statement. These signs make the seventeen- and twenty-circle roots unique. The sixteen-circle proof uses continuity and later obtains root uniqueness from optimality.

A.3 Bounds used in the geometric reductions

For each row of Table 14, the geometric computation in the corresponding case section certifies

$$b_- < b_N < b_+, \quad c(b_N) < c_+, \quad c_+^2 < b_-^2 - 4, \quad r_*^- < r_*(b_N). \quad (\text{A.1})$$

It also certifies the four corner inequalities (2.14), with r_*^- and c_+ as their endpoints. Thus the angular length bound applies to the projected inner set, and h_b and f_b remain nondecreasing throughout the larger interval $[0, c_+]$ used for the count exclusions.

N	b_-	b_+	c_+	r_*^-
15	3.5213569	3.521357	2.386	1.64
16	3.615425	3.615426	2.510	1.7
17	3.792033	3.792034	2.738	1.9
18	3.863703305	3.863703306	2.828428	1.9999999
20	4.122320736	4.122320737	3.151994	2.26714

Table 14: Exact rational inputs to the common geometric tests. The lower and upper bounds have the directions specified in (A.1).

The same computations give Table 15. For example, to prove a lower endpoint η_- , compare a rational upper bound for $\sin \eta_-$ with $1/b_+$. To prove an upper endpoint η_+ , compare a lower bound for $\sin \eta_+$ with $1/b_-$. The sine function is increasing on the small positive intervals in this table, so these comparisons have the required directions.

N	$\eta_- < \eta(b_N)$	$\eta_+ > \eta(b_N)$
15	.287943	.287944
16	.280246539	.280246620
17	.266867080	.266867153
18	.261799387740	.261799387810
20	.245026263334	.245026263395

Table 15: Exact rational lower and upper bounds, in radians.

To obtain the preliminary counts, use (2.8) for the outer set and Lemma 2.8 for the inner set. In the latter lemma the second cutoff is $d = c_+ - 4/c_+$. The intermediate counts are displayed in Table 16, so each total can be checked by addition. The small-disk conditions are $d < 1$ for fifteen and sixteen, $d^2 < 2$ for seventeen, $d < 3/2$ for eighteen and $d < 2$ for twenty. For eighteen the additional chord comparison is $2(3/2)^2(1 - \cos(2\pi/5)) < 4$, as required by Lemma 2.7 with $\mu = 5$.

N	At radius c_+	Inside d	k at most	m at most	Permitted m
15	7	1	8	10	7–10
16	7	1	8	11	8–11
17	8	3	11	11	6–11
18	8	4	12	12	6–12
20	9	5	14	12	6–12

Table 16: Exact preliminary count bounds. The second and third columns refer to the auxiliary projection of the inner set in Lemma 2.8; their sum bounds k , and $m = N - k$.

These bounds give a finite range of counts to consider. Appendix C records the subsequent exclusions and localizations, including their completed subdivision counts.

B Exact interval arithmetic and angle bounds

Proof strategy. A numerical test must hold for every real value in its input intervals. We obtain that guarantee by enclosing each intermediate value between rational endpoints. We first give the arithmetic rules, then justify the trigonometric remainders and the comparisons that certify an angle. These rules supply every numerical premise of the computations in the main text.

B.1 Arithmetic operations and radicals

Let $x_{\text{int}} \in [\iota_-, \iota_+]$ and $y_{\text{int}} \in [j_-, j_+]$, where all four endpoints are rational. Addition and multiplication satisfy

$$\begin{aligned} x_{\text{int}} + y_{\text{int}} &\in [\iota_- + j_-, \iota_+ + j_+], \\ x_{\text{int}} y_{\text{int}} &\in \left[\min_{a,d \in \{-,+\}} \iota_a j_d, \max_{a,d \in \{-,+\}} \iota_a j_d \right]. \end{aligned} \quad (\text{B.1})$$

The multiplication rule follows by maximizing or minimizing a linear function in each argument in turn. If the interval for x_{int} excludes zero, its reciprocal lies in $[1/\iota_+, 1/\iota_-]$, because the reciprocal function decreases on each side of zero. Division uses this reciprocal interval. For x_{int}^2 , the upper endpoint is $\max(\iota_-^2, \iota_+^2)$; the lower endpoint is zero if the interval contains zero and otherwise is the minimum of those squares.

Every decimal input denotes an exact rational number. When an operation is evaluated on a fixed grid of denominator Q_{grid} , a lower endpoint is rounded down and an upper endpoint up. Consequently each operation preserves enclosure, even after many successive evaluations. Arbitrarily large integers suffice to implement these rules exactly.

For a nonnegative rational x_{int} , choose the integer j_{sqrt} satisfying

$$j_{\text{sqrt}}^2 \leq Q_{\text{grid}}^2 x_{\text{int}} < (j_{\text{sqrt}} + 1)^2.$$

Then $j_{\text{sqrt}}/Q_{\text{grid}} \leq \sqrt{x_{\text{int}}} < (j_{\text{sqrt}} + 1)/Q_{\text{grid}}$. Applying this construction to the endpoints bounds a square root on an interval. Its lower endpoint may be replaced by zero when the radicand is known to be nonnegative and the enclosure extends below zero.

For a differentiable construction, carry an interval for the value and an interval for its derivative together. The sum, product and quotient rules give enclosing derivative intervals by the same arithmetic operations. In particular, $(\sqrt{F})' = F'/(2\sqrt{F})$ is used only when the radicand has a strictly positive lower bound. The resulting continuous functions have a root between endpoints of opposite signs. A strict derivative sign throughout that interval proves that the root is unique there.

B.2 Trigonometric series and their remainders

The sine and cosine bounds come from the expansions

$$\cos z = \sum_{j=0}^{\infty} \frac{(-1)^j z^{2j}}{(2j)!}, \quad \sin z = \sum_{j=0}^{\infty} \frac{(-1)^j z^{2j+1}}{(2j+1)!}.$$

Taylor's theorem proves these identities: every derivative has absolute value at most 1, so the remainder after degree n_{deg} is at most $|z|^{n_{\text{deg}}+1}/(n_{\text{deg}}+1)!$, which tends to zero for fixed z . For $0 \leq z \leq 355/113$, the absolute values of all omitted terms after degree 25 decrease. Indeed, the ratio of consecutive nonzero terms is $z^2/((n_{\text{deg}}+1)(n_{\text{deg}}+2)) < 1$ once $n_{\text{deg}} \geq 25$. The alternating tail therefore lies between zero and its first term, with that term's sign. A partial sum ending with a negative term bounds below; one ending with a positive term bounds above.

This proves the sine bounds of degrees 25, 27, 29, 37 and cosine bounds of degrees 26, 34 used in the arithmetic tests.

For arctangent, the finite geometric identity gives

$$\frac{1}{1+z^2} = \sum_{j=0}^{n_{\text{terms}}-1} (-1)^j z^{2j} + \frac{(-1)^{n_{\text{terms}}} z^{2n_{\text{terms}}}}{1+z^2}.$$

Integrating from 0 to $z \geq 0$ and bounding the last denominator below by 1 yields

$$0 \leq (-1)^{n_{\text{terms}}} \left(\arctan z - \sum_{j=0}^{n_{\text{terms}}-1} \frac{(-1)^j z^{2j+1}}{2j+1} \right) \leq \frac{z^{2n_{\text{terms}}+1}}{2n_{\text{terms}}+1}. \quad (\text{B.2})$$

Here n_{terms} is any positive integer. This bound includes the finite sums with twenty, thirty and forty terms used to enclose π ; each evaluation is an exact rational calculation.

B.3 Certifying a lower bound for an angle

Suppose $r, s > 0$ and $|r - s| < 2 < r + s$. Since cosine decreases on $[0, \pi]$, a rational number z satisfies $z \leq g(r, s)$ if

$$0 \leq z \leq \pi, \quad \cos z \geq \frac{r^2 + s^2 - 4}{2rs}. \quad (\text{B.3})$$

The lower cosine polynomial

$$P_{\cos,26}(z) = \sum_{j=0}^{13} \frac{(-1)^j z^{2j}}{(2j)!} \leq \cos z \quad (0 \leq z \leq 355/113) \quad (\text{B.4})$$

therefore suffices for the second comparison. The analogous polynomial $P_{\cos,34}$, summed through $j = 17$, is used for separately supplied angle tables. Both end with a negative term, so the remainder argument above proves the stated direction.

If the radii are enclosed in intervals, bound the quotient in (B.3) above by interval arithmetic. In particular, division retains the numerator's sign: for a negative numerator an upper quotient bound uses the larger positive denominator. Comparing a cosine lower bound with the quotient upper bound then proves the inequality for every enclosed radius. In the four-corner test this comparison is made at each endpoint pair, after which Lemma 4.1 gives the bound throughout the radial rectangle. The degenerate branches of g in (2.6) are handled directly before division.

B.4 A rational upper bound for one full turn

We obtain bounds on π from Machin's identity

$$\pi = 16 \arctan(1/5) - 4 \arctan(1/239). \quad (\text{B.5})$$

For a verification, put $\alpha_M = \arctan(1/5)$ and $\beta_M = \arctan(1/239)$. Tangent addition gives $\tan(2\alpha_M) = 5/12$, $\tan(4\alpha_M) = 120/119$, and $\tan(4\alpha_M - \beta_M) = 1$. Because $0 < 4\alpha_M - \beta_M < 4/5 < \pi/2$, this last angle is $\pi/4$, proving the identity.

Applying (B.2) to its two terms gives

$$3.141592653589793238 < \pi < 3.141592653589793239. \quad (\text{B.6})$$

For example, use forty terms in each arctangent sum, with the first omitted term as its upper error bound; substitution gives these strict rational comparisons. The full-order and position calculations use the common scale and full-turn bound

$$Q_\theta = 2^{40} = 1099511627776, \quad \Pi^+ = \frac{6908435304716}{Q_\theta} > 2\pi. \quad (\text{B.7})$$

The last inequality follows by comparison with twice the upper endpoint in (B.6). Thus both the forward angle bounds and the subtracted full-turn bound have certified rounding directions.

C Inputs and records of the finite subdivisions

The computations in the case sections specify which tests can exclude a box and which target ranges can accept it. We record the initial boxes and orders here, followed by their completed counts. Proposition 4.2 supplies the logical link: if every final box is resolved, every feasible set in the initial ranges has one of the accepted matches.

C.1 Excluding inner counts

For each row of Table 17, start with $[0, c_+]^k$ for every listed k . For twenty circles, round the upper endpoint outward to denominator 2^{30} . Use the inner cyclic order and the labeling inequalities (4.2): r_1 is a minimum radius and $r_2 \leq r_k$. The count computations apply the tests specified in their statements. They accept no terminal boxes; success requires that every terminal box be excluded.

N	c_+	Excluded k	Remaining m	Proposition
15	2.386	6, 7, 8	10	7.3
16	2.510	6, 7, 8	11	8.3
17	2.738	8, 9, 10, 11	10, 11	9.3
18	2.828428	9, 10, 11, 12	10, 11, 12	10.2
20	3.151994	9, 10, 11, 12, 13, 14	12	11.3

Table 17: Exact inputs and conclusions of the inner-count exclusions. Each value of c_+ is an exact rational upper bound on the cutoff.

The individual completed counts appear in Table 18. Every terminal box is excluded by either the angular-length test (4.10) or the pair-angle tests, after any valid radius contraction. The two terminal counts sum to half the total box count plus one half, as the subdivision-tree identity below requires.

C.2 Full cyclic orders and their starting boxes

For a full order, the nonnegative integers $q_1, \dots, q_k$ sum to m . In the fifteen- and sixteen-circle cases, the one-gap lemma requires each q_i to be positive. Generate every such integer list, and retain the lexicographically least list among its cyclic shifts and reversals. Section 4.2 justifies this reduction because the accompanying radial box is unchanged by all those relabelings.

Table 19 records every starting case. The count of positive lists is $\binom{m-1}{k-1}$; the count of nonnegative lists is $\binom{m+k-1}{k-1}$. These follow by placing $k-1$ dividers between m objects, or between the m objects and $k-1$ additional divider positions, respectively. Direct generation of the lists and selection of their least representatives gives the orbit counts in Table 20.

The target ranges are those in Propositions 7.4, 8.4, 9.4, 10.4 and 11.4. For fifteen, all five intervals must lie in $[1.693, 1.709]$ and the count list must be $(2, 2, 2, 2, 2)$. For sixteen and seventeen, each interval must be contained in its specified target range, with the specified count list. For eighteen, every choice of a near-origin center is tested, and the remaining intervals and merged outer counts must match one of the six references. For twenty, each of the eight choices of an omitted inner center is tested; the seven retained intervals and their merged counts must meet the stated target conditions. Every original center participates in the exclusion tests before these final matching operations.

Stored radial endpoints have denominator 2^{30} for the fifteen- and sixteen-circle full-order computations, and 2^{40} for the other full-order computations. Initial endpoints are rounded outward. Stored angle bounds have denominator $Q_\theta = 2^{40}$, as in (B.7). Each radial interval

N	k	Boxes examined	Length exclusions	Angle exclusions
15	6	1,819	394	516
15	7	1,149	242	333
15	8	81	5	36
16	6	1,463	352	380
16	7	2,599	556	744
16	8	817	114	295
17	8	100,475	23,391	26,847
17	9	4,585	481	1,812
17	10	1,267	22	612
17	11	777	0	389
18	9	12,733	1,726	4,641
18	10	6,281	463	2,678
18	11	2,273	31	1,106
18	12	1,035	0	518
20	9	396,147	82,039	116,035
20	10	320,407	51,626	108,578
20	11	217,719	26,231	82,629
20	12	109,203	9,425	45,177
20	13	16,371	270	7,916
20	14	3,799	0	1,900

Table 18: Exact completed counts for the inner-count exclusions. All terminal boxes are excluded; none is accepted or unresolved.

N	m	k	Initial radial box	Allowed q_i	All lists
15	10	5	$[\cdot 281, 2.221]^5$	positive	126
16	11	5	$[\cdot 242, 1.812]^5$	positive	210
17	11	6	$[0, 2.368]^6$	nonnegative	4,368
17	10	7	$[0, 2.738]^7$	nonnegative	8,008
18	11	7	$[0, 2.828427124746191]^7$	nonnegative	12,376
18	10	8	$[0, 2.828427124746191]^8$	nonnegative	19,448
20	12	8	$[0, 2.806]^8$	nonnegative	50,388

Table 19: Rational starting ranges for the full-order computations. The implemented boxes round these endpoints outward to the radial denominators stated below. Each box therefore contains its listed range.

represents every real radius between its endpoints. When a box is bisected, both children include the splitting endpoint and their union covers the parent. A smallest interval that satisfies neither an exclusion test nor an acceptance condition causes failure. Every recorded calculation finishes with all cases resolved.

N	m	Order orbits	Boxes examined	Accepted boxes	Angle bounds
15	10	16	63,846	175	1,307
16	11	26	44,580	190	4,805
17	11	392	107,250	0	2,353
17	10	600	15,297,824	8,423	159,816
18	11	912	589,172	524	6,340
18	10	1,282	1,299,422	100	3,290
20	12	3,260	5,822,866	5,406	26,878

Table 20: Exact completed counts. An accepted box is contained in a specified target range and may represent infinitely many real choices. Every supplied angle bound is validated by rational arithmetic.

There is also a check on each subdivision tree. Let n_{split} be its number of split boxes and n_{leaf} its number of terminal boxes. Each split has two children, and every box except the initial one has one parent. Thus $2n_{\text{split}} = n_{\text{split}} + n_{\text{leaf}} - 1$, so the total number of boxes is $2n_{\text{leaf}} - 1$. This identity checks the completed records for each starting order. Coverage itself follows from the enclosing intervals, the justified tests and the union property at every split, as proved in Proposition 4.2.

D Worked bounds for individual positions

The position computations bound both an unknown center set and its reference set inside the same relative-angle intervals. After one outer direction is aligned, the width of such an interval bounds the difference of the two directions. We give one full calculation for each of the sixteen-, seventeen- and twenty-circle cases. Each uses two chains of directed inequalities to obtain the two endpoints, followed by the polar position estimate.

In this appendix, $\vartheta_{\mathbf{p}}$ denotes the chosen angle of the unknown center matched to a reference center $\mathbf{p}$. The reference angles satisfy the same bounds. Each subsection adopts the point labels of its corresponding case section. Within that subsection, ℓ_j is the pair-angle lower bound in row j of its table. All tables use the exact scale Q_θ and full-turn upper bound Π^+ from (B.7).

D.1 Sixteen circles

We bound the displacement of the center matched to $\mathbf{SA}$, using $\mathbf{Q}_0 = (0, -b_{16})$ as the aligned reference. The radial intervals are those in (8.8), and the cyclic order is (8.9). The pair bounds needed for the two chains appear in Table 21.

j	Reference pair	$Q_\theta \ell_j$
1	$\mathbf{Q}_0, \mathbf{Q}_1$	616268729228
2	$\mathbf{Q}_5, \mathbf{Z}$	372873202735
3	$\mathbf{Z}, \mathbf{M}$	1322619375527
4	$\mathbf{M}, \mathbf{A}$	1405480958256
5	$\mathbf{A}, \mathbf{SA}$	1452229930672

Table 21: Exact numerators of rational pair-angle lower bounds. The first row applies to any two outer centers. Reflected pairs have the same radial intervals and the same bounds.

For example, both radii in the last row belong to $[1.63017321, 1.63017521]$. At each of its four endpoint pairs, the rational polynomial test of Appendix B verifies

$$\cos\left(\frac{1452229930672}{Q_\theta}\right) \geq \frac{r^2 + s^2 - 4}{2rs}.$$

Lemma 4.1 extends the lower angle bound from those endpoints to the full rectangle. The other rows are certified by the same specified test in Proposition 8.5.

The first chain has five outer-outer steps and three further pair types. Their total lower bound is

$$\Lambda := 5\ell_1 + \ell_2 + \ell_3 + \ell_4 = \frac{6182317182658}{Q_\theta}.$$

The chain itself is

$$\mathbf{Q}_0, \mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3, \mathbf{Q}_4, \mathbf{Q}_5, \mathbf{Z}, \mathbf{SM}, \mathbf{SA}.$$

Its step from $\mathbf{Q}_1$ to $\mathbf{Q}_2$ returns to an earlier entry of (8.9); every other step follows that list. Subtracting Π^+ once, as prescribed by (4.3), gives

$$\vartheta_{\mathbf{SA}} - \vartheta_{\mathbf{Q}_0} \geq \Lambda - \Pi^+. \quad (\text{D.1})$$

For the opposite direction use the chain

$$\mathbf{A}, \mathbf{M}, \mathbf{Z}, \mathbf{SQ}_5, \mathbf{SQ}_4, \mathbf{SQ}_3, \mathbf{SQ}_2, \mathbf{SQ}_1, \mathbf{Q}_0.$$

All its steps follow the list, so it gives

$$\vartheta_{\mathbf{Q}_0} - \vartheta_{\mathbf{A}} \geq \Lambda. \quad (\text{D.2})$$

Add the reverse pair bound $\vartheta_{\mathbf{A}} - \vartheta_{\mathbf{SA}} \geq \ell_5 - \Pi^+$ to (D.2). Combining the result with (D.1) yields

$$-\frac{726118122058}{Q_\theta} \leq \vartheta_{\mathbf{SA}} - \vartheta_{\mathbf{Q}_0} \leq -\frac{726111808614}{Q_\theta}.$$

The width is $6313444/Q_\theta < 5.742044 \cdot 10^{-6}$. Both endpoints are negative because the chosen angle of $\mathbf{SA}$ precedes that of $\mathbf{Q}_0$. Both the unknown and reference set use this same ordering, so the width bounds the difference of their relative angles.

The shared radial interval has width $2 \cdot 10^{-6}$. After alignment, Lemma 5.3 therefore gives

$$\|\mathbf{x}_{\mathbf{SA}} - \mathbf{SA}\| \leq 2 \cdot 10^{-6} + b_{16} \frac{6313444}{Q_\theta} < 2.276 \cdot 10^{-5},$$

where the last comparison uses $b_{16} < 3.615425595$. This is below the local threshold $1/10000$.

D.2 Seventeen circles: two reference directions coincide

This subsection uses the point labels and local notation of Section 9. We give the bound for the center matched to $\mathbf{G}$, which has the largest position error bound. The reference centers $\mathbf{G}$ and $\mathbf{K}$ lie on one ray; the unknown centers retain their order but may have distinct directions. The calculation keeps that distinction explicit through a zero lower bound for their ordered angular difference.

j	Reference pair	$Q_\theta \ell_j$
1	$\mathbf{A}, \mathbf{C}$	586846955351
2	$\mathbf{C}, \mathbf{E}$	313137629044
3	$\mathbf{E}, \mathbf{G}$	2260807738297
4	$\mathbf{K}, \mathbf{J}$	960343861311
5	$\mathbf{J}, \mathbf{M}$	400480862097
6	$\mathbf{G}, \mathbf{K}$	0

Table 22: Exact numerators of the lower angle bounds used for seventeen circles. Row 1 applies to any outer pair; rows 2 and 5 apply to the indicated inner center and any outer center. The bounds also apply to reflected pairs.

Proposition 9.5 verifies the bounds in Table 22 over the full radial intervals (9.12). Use the order (9.13) and align at $\mathbf{SA}$. The chain

$$\mathbf{SA}, \mathbf{A}, \mathbf{C}, \mathbf{E}, \mathbf{G}$$

returns to an earlier entry only at its last step. Thus

$$\vartheta_{\mathbf{G}} - \vartheta_{\mathbf{SA}} \geq 2\ell_1 + \ell_2 + \ell_3 - \Pi^+ = -\frac{3160796026673}{Q_\theta}.$$

In the other direction, every step of the chain

$$\mathbf{G}, \mathbf{K}, \mathbf{SJ}, \mathbf{SM}, \mathbf{SH}, \mathbf{SE}, \mathbf{SC}, \mathbf{SA}$$

follows the chosen list. Its first step uses $\vartheta_{\mathbf{K}} - \vartheta_{\mathbf{G}} \geq 0$, and the whole chain gives

$$\begin{aligned} \vartheta_{\mathbf{SA}} - \vartheta_{\mathbf{G}} &\geq \ell_6 + \ell_4 + \ell_5 + 2\ell_2 + 2\ell_1 \\ &= \frac{3160793892198}{Q_\theta}. \end{aligned}$$

The two chains therefore enclose the relative angle in

$$-\frac{3160796026673}{Q_\theta} \leq \vartheta_{\mathbf{G}} - \vartheta_{\mathbf{SA}} \leq -\frac{3160793892198}{Q_\theta}.$$

Its width is $2134475/Q_\theta$. The radial width is $2 \cdot 10^{-7}$, so the polar position estimate gives

$$\|\mathbf{x}_{\mathbf{G}} - \mathbf{G}\| \leq 2 \cdot 10^{-7} + 3.792033749 \cdot \frac{2134475}{Q_\theta} < 7.562 \cdot 10^{-6}.$$

Here $3.792033749 > b_{17}$ follows from (9.1). The resulting bound is strictly below the local threshold $1/16000$.

D.3 Twenty circles: a bound for the retained set

This subsection uses the point labels and local notation of Section 11. The largest position bound among the nineteen retained centers occurs at $\mathbf{F}$ and its reflection. We derive it for $\mathbf{F}$ by two directed chains in (11.9). The pair bounds in Table 23 are certified over the full radial intervals by Proposition 11.5 and Lemma 4.1.

j	Reference pair	$Q_\theta \ell_j$
1	$\mathbf{SA}, \mathbf{A}$	538818451292
2	$\mathbf{A}, \mathbf{F}$	240544847290
3	$\mathbf{F}, \mathbf{E}$	1030220889616
4	$\mathbf{E}, \mathbf{J}$	1046670915227
5	$\mathbf{J}, \mathbf{K}$	867347980931
6	$\mathbf{K}, \mathbf{L}$	490714158647

Table 23: Exact numerators of lower angle bounds for twenty circles. The first row applies to any outer pair, and the last row also applies to $\mathbf{K}, \mathbf{SL}$.

The short chain $\mathbf{SA}, \mathbf{A}, \mathbf{F}$ follows the list and gives

$$\vartheta_{\mathbf{F}} - \vartheta_{\mathbf{SA}} \geq \ell_1 + \ell_2 = \frac{779363298582}{Q_\theta}.$$

For the other direction use the chain

$$\mathbf{F}, \mathbf{E}, \mathbf{J}, \mathbf{K}, \mathbf{SL}, \mathbf{SU}, \mathbf{SM}, \mathbf{SH}, \mathbf{SC}, \mathbf{SA}.$$

Only its step from $\mathbf{E}$ to $\mathbf{J}$ returns to an earlier entry. There are five outer–outer steps, so

$$\begin{aligned} \vartheta_{\mathbf{SA}} - \vartheta_{\mathbf{F}} &\geq \ell_3 + \ell_4 + \ell_5 + \ell_6 + 5\ell_1 - \Pi^+ \\ &= -\frac{779389103835}{Q_\theta}. \end{aligned}$$

Combining the two inequalities yields

$$\frac{779363298582}{Q_\theta} \leq \vartheta_{\mathbf{F}} - \vartheta_{\mathbf{SA}} \leq \frac{779389103835}{Q_\theta}.$$

The width is $25805253/Q_\theta$. Since the radial interval has width $2 \cdot 10^{-5}$ and $b_{20} < 4.122320737$, alignment at **SA** gives

$$\|\mathbf{x}_F - \mathbf{F}\| \leq 2 \cdot 10^{-5} + 4.122320737 \cdot \frac{25805253}{Q_\theta} < .00011675.$$

The final comparison is between exact rationals. Together with (11.10), it places the displacement strictly below the local threshold $1/8400$.

E Derivative estimates for fifteen circles

Proof strategy. This appendix uses the point labels and local notation of Section 7. Lemma 7.5 needs the linear terms and uniform second-derivative bounds for Φ and Ψ . We compute the linear terms at the construction, then bound the second derivatives over the full allowed square. The proof in the main text applies Taylor's theorem to these bounds.

Recall that $\varphi = \pi/5$, $a = 1/\sin \varphi$, $t = \cot \varphi$, $\chi = 1/t$ and $b_{15} = \sqrt{(t+2)^2 + 1}$. The functions are

$$\Phi(\xi, \zeta) = g(a(1 + \xi), a(1 + \zeta)), \quad \Psi(\xi) = h(a(1 + \xi)),$$

where $|\xi|, |\zeta| \leq 1/200$. The rational comparisons $2.23606^2 < 5 < 2.23607^2$ and $a^2 = 2 + 2/\sqrt{5}$ give $1.70130 < a < 1.70131$, and hence

$$1.692 < a(1 - 1/200) < 1.693, \quad 1.709 < a(1 + 1/200) < 1.711.$$

Thus both varying radii lie in $(1.692, 1.711)$. Two such radii satisfy $|r - s| < .019 < 2 < r + s$. A radius in this interval and $b_{15} \in (3.5213569, 3.521357)$ also satisfy $|b_{15} - r| < 1.829357 < 2 < r + b_{15}$. Lemma 5.1 therefore applies throughout both domains.

E.1 The linear terms

Claim E.1. *At the unperturbed radii, the values and first derivatives are*

$$\begin{aligned} \Phi(0, 0) &= 2\varphi, & (\Phi)_\xi(0, 0) &= (\Phi)_\zeta(0, 0) = -\chi, \\ \Psi(0) &= \varphi - \eta(b_{15}), & \Psi'(0) &= t. \end{aligned} \tag{E.1}$$

Proof. The angle values follow from the contact identities (7.6). For the derivatives of Φ , put $r = s = a$ in (5.2). Then $\mathcal{D}(r, s) = 16(a^2 - 1) = 16t^2$, and multiplication by $dr/d\xi = a$ gives $(\Phi)_\xi(0, 0) = -1/t$. Exchanging the two equal arguments gives the same derivative in ζ .

For Ψ , put $r = a$ and $s = b_{15}$. The identities $a^2 = t^2 + 1$ and $b_{15}^2 = t^2 + 4t + 5$ give $r^2 - s^2 + 4 = -4t$ and $\mathcal{D}(r, s) = 16$. Formula (5.2), multiplied by a , therefore gives $\Psi'(0) = t$, completing the four assertions. $\square$

E.2 The second derivatives

To bound the denominators in (5.3)–(5.4), use the exact rational squared-radius bounds

$$\alpha^- = 1.692^2, \quad \alpha^+ = 1.711^2, \quad \beta^- = 3.5213569^2, \quad \beta^+ = 3.521357^2. \tag{E.2}$$

They satisfy $\alpha^- < r^2, s^2 < \alpha^+$ for two varying inner radii, and $\beta^- < b_{15}^2 < \beta^+$. The next two claims bound the inner–inner and inner–outer expressions separately.

Claim E.2. *Throughout $|\xi|, |\zeta| \leq 1/200$,*

$$|(\Phi)_{\xi\xi}| + |(\Phi)_{\xi\zeta}| < 4, \quad |(\Phi)_{\zeta\zeta}| + |(\Phi)_{\xi\zeta}| < 4.$$

Proof. Proof strategy. We first bound the discriminant below to control its square root. The formulas in Lemma 5.1 then bound each numerator above. Finally, the chain-rule factors convert derivatives in radii to derivatives in the proportional changes ξ, ζ .

Because $2\alpha^- - 4 > 0$, the discriminant satisfies

$$\mathcal{D}(r, s) \geq 4(\alpha^-)^2 - (2\alpha^+ - 4)^2 =: \mathcal{D}_{\text{in}}^- > 25. \quad (\text{E.3})$$

Thus $\sqrt{\mathcal{D}} > 5$. The terms with denominator $\mathcal{D}^{3/2}$ in $|r^2 g_{rr}| + |rs g_{rs}|$ have sum at most

$$M_1 := \frac{16\alpha^+(2\alpha^+ - 4) + 32(\alpha^+)^2}{5\mathcal{D}_{\text{in}}^-}.$$

The remaining term in (5.3) is bounded in absolute value by $M_2 := (\alpha^+ - \alpha^- + 4)/5$. Since $(\Phi)_{\xi\xi} = a^2 g_{rr}$ and $(\Phi)_{\xi\zeta} = a^2 g_{rs}$, while $r/a, s/a \geq 199/200$, we obtain

$$\begin{aligned} |(\Phi)_{\xi\xi}| + |(\Phi)_{\xi\zeta}| &= \frac{a^2}{r^2} |r^2 g_{rr}| + \frac{a^2}{rs} |rs g_{rs}| \\ &\leq \frac{M_1 + M_2}{(199/200)^2} < 4 < 20. \end{aligned} \quad (\text{E.4})$$

The last comparison is rational after substitution from (E.2). Exchanging r, s leaves all these bounds unchanged and gives the asserted estimate for the other row. $\square$

Claim E.3. *For every $|\xi| \leq 1/200$, we have $|\Psi''(\xi)| < 15$.*

Proof. Put $s = b_{15}$ in (5.1). Since $\alpha^- + \beta^- - 4 > 0$, its lower bound is

$$\mathcal{D}(r, b_{15}) \geq 4\alpha^- \beta^- - (\alpha^+ + \beta^+ - 4)^2 =: \mathcal{D}_{\text{out}}^- > 9. \quad (\text{E.5})$$

Thus $\sqrt{\mathcal{D}} > 3$. Also $r^2 - b_{15}^2 + 4 < 0$ throughout the interval. The two terms in (5.3) therefore give the bounds

$$M_3 := \frac{16\alpha^+(\alpha^+ + \beta^+ - 4)}{3\mathcal{D}_{\text{out}}^-}, \quad M_4 := \frac{\beta^+ - \alpha^- - 4}{3}.$$

Multiplying by $a^2/r^2 \leq (200/199)^2$ yields

$$|\Psi''(\xi)| \leq \frac{M_3 + M_4}{(199/200)^2} < 14.918 < 15 < 20. \quad (\text{E.6})$$

Again the final comparison uses only the rational values in (E.2), proving the claim. $\square$

Claims E.1–E.3 supply the values and derivative bounds used in (7.20)–(7.21). The segment calculation (7.22) and Lemma 5.2 then give both lower angle estimates of Lemma 7.5.

F Derivative bounds for eighteen circles

Proof strategy. This appendix uses the point labels and local notation of Section 10. Proposition 10.6 controls the change of each contact angle when an interior radius changes by at most 10^{-6} . We give rational estimates for all three types of pair in Table 9. The first estimate is written out, and a table records the corresponding numerator and denominator bounds for the other two.

Use $3.863703 < b_{18} < 3.863704$ and $2.828427124746190 < c(b_{18}) < 2.828427124746191$. The second interval follows by squaring its endpoints and comparing with 8, since $c(b_{18}) = 2\sqrt{2}$. Thus the required radius intervals are enclosed by

$$I_2 = [1.999999, 2.000001], \quad I_c = [2.828426124746190, 2.828428124746191],$$

and the boundary radius is enclosed by $I_b = [3.863703, 3.863704]$. These three symbols denote only the intervals in this appendix. Each pair of intervals used below has positive endpoints and satisfies $|r - s| < 2 < r + s$, so Lemma 5.1 applies on the whole rectangle.

For $r \in I_2$ and $s = b_{18} \in I_b$, the discriminant has lower bound

$$\begin{aligned} \mathcal{D}(r, b_{18}) &\geq 4(1.999999)^2(3.863703)^2 \\ &\quad - ((2.000001)^2 + (3.863704)^2 - 4)^2 \\ &> 15.99 > 3.998^2. \end{aligned}$$

The expression $b_{18}^2 - r^2 - 4$ is positive throughout this interval. Formula (5.2) therefore gives

$$|h'(r)| < \frac{(3.863704)^2 - (1.999999)^2 - 4}{1.999999 \cdot 3.998} < .867.$$

Every term in the last comparison is rational.

For completeness, Table 24 gives bounds for all three rows. If $r \in [r_-, r_+]$ and $s \in [s_-, s_+]$ in one of these rectangles, then $r^2 + s^2 - 4 > 0$ and

$$\mathcal{D}(r, s) \geq 4r_-^2 s_-^2 - (r_+^2 + s_+^2 - 4)^2.$$

Also $|r^2 - s^2 + 4|$ is bounded above by the larger absolute endpoint of $[r_-^2 - s_+^2 + 4, r_+^2 - s_-^2 + 4]$. These two rules give the middle columns of the table. Dividing the numerator bound by the lower radius times the lower square-root bound gives the derivative bound through (5.2).

r	s	$\mathcal{D}$ lower	$\sqrt{\mathcal{D}}$ lower	Numerator upper
I_2	I_b	15.99	3.998	6.928213
I_c	I_b	119.424	10.928	2.928215
I_2	I_2	47.9998	6.928	4.000008

Table 24: Exact rational bounds for the derivative comparisons. The discriminant and square-root bounds are strict lower bounds; the last column bounds $|r^2 - s^2 + 4|$ above, non-strictly.

The resulting strict derivative upper bounds are .867, .095 and .289, respectively. In the last rectangle, exchanging r, s preserves the domain, so the same bound holds for g_s . This verifies every derivative estimate used in the gap comparison.

G Contact and matrix certificate data

The local computations require a definite ordering of coordinates and contact rows, because the rational matrix P must multiply the corresponding matrix H . We specify that ordering first. We then describe the rational input data and the exact acceptance inequalities from Lemma 5.6.

G.1 Ordering the centers and contact rows

For the sixteen-, seventeen- and twenty-circle references, index the centers from zero in the order of Table 25. The twenty-circle column indexes only the nineteen points of $\mathcal{C}$. In each column the fixed horizontal coordinate is that of center 0.

For each eighteen-circle reference, center 0 is $(0, -b_{18})$ and center 1 is O . The remaining centers are appended in the order shown in Table 26, reading across each row. An entry j in a radius column means the center $re(j\pi/12)$ with that column's radius r . Thus the table uses integer multiples of $\pi/12$ throughout, and all coordinate formulas remain exact.

Index	$\mathcal{P}_{16}$	$\mathcal{P}_{17}$	$\mathcal{C}$
0	Q₀	SA	SA
1	SQ₁	A	A
2	Q₁	SC	SC
3	SQ₂	C	C
4	Q₂	F	SF
5	SA	SE	F
6	A	E	SH
7	SQ₃	SH	H
8	Q₃	H	SE
9	SM	G	E
10	M	SM	SM
11	Z	M	M
12	SQ₄	SJ	SJ
13	Q₄	J	J
14	SQ₅	K	SU
15	Q₅	SL	U
16		L	K
17			SL
18			L

Table 25: Coordinate order for the local matrices. Blank entries lie beyond the size of the corresponding reference set.

Reference	Radius 2	Radius $c(b_{18})$	Radius b_{18}
$\mathcal{Q}_1$	13, 17, 21, 1, 7	4, 10	12, 14, 16, 20, 22, 0, 2, 6, 8
$\mathcal{Q}_2$	13, 17, 23, 3, 9	20, 6	12, 14, 16, 22, 0, 2, 4, 8, 10
$\mathcal{Q}_3$	13, 17, 21, 5, 9	2	12, 14, 16, 20, 22, 0, 4, 6, 8, 10
$\mathcal{Q}_4$	13, 17, 21, 3, 9	6	12, 14, 16, 20, 22, 0, 2, 4, 8, 10
$\mathcal{Q}_5$	13, 17, 21, 3, 7	10	12, 14, 16, 20, 22, 0, 2, 4, 6, 8
$\mathcal{Q}_6$	1, 5, 9, 13, 17, 21	—	0, 6, 8, 10, 12, 14, 16, 20, 22

Table 26: Exact angular indices for the matrix coordinate order of the six eighteen-circle references, after centers 0 and 1.

Select all the pair contacts specified in Propositions 8.1, 9.1, 10.3 and 11.1, and all container contacts. For an eighteen-circle reference, enumerate all index pairs $i < j$ and select exactly those with squared distance 4. Proposition 10.3 checks these equalities coefficient by coefficient in $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ and checks strict separation for the other pairs. Thus the selection of its contact rows is exact.

List pair rows first in lexicographic order of (i, j) , then boundary rows in increasing index order. List horizontal and vertical coordinates for center 0, then center 1, and so on, and delete the first horizontal coordinate. These conventions and the row formulas in (5.17) specify the signed rigidity matrix **A** completely. Its augmentation $\mathbf{H} = [\mathbf{A} \mid \mathbf{1} \mid \mathbf{V}]$ has the dimensions in Table 27.

G.2 Rational input matrices and acceptance conditions

The input for each reference consists of an integer square array and a positive integer scale Q_P ; entrywise division gives the proposed inverse **P**. When extra columns are required, an integer rectangular array and positive scale Q_V give **V** in the same way. Every inverse scale is $Q_P = 10^{15}$. The extra-column scale is 10^{12} for $\mathcal{P}_{17}$ and $\mathcal{Q}_6$, and 10^{14} for $\mathcal{Q}_1, \dots, \mathcal{Q}_5$. The

other two matrices have no extra columns. The archived arrays and their correspondence to computations are identified in Appendix H.

Reference	n_{ref}	n_{var}	n_{con}	Extra columns	κ	σ_*
$\mathcal{P}_{16}$	16	31	32	0	5	1/250
$\mathcal{P}_{17}$	17	33	35	1	8	1/250
$\mathcal{Q}_1, \mathcal{Q}_2$	18	35	44	8	3	1/64
$\mathcal{Q}_3, \mathcal{Q}_4, \mathcal{Q}_5$	18	35	43	7	3	1/100
$\mathcal{Q}_6$	17	33	41	7	4	1/70
$\mathcal{C}$	19	37	38	0	7	1/150

Table 27: Exact matrix dimensions and certified constants. The absolute row sums of $\mathbf{H}^{-1}$ are strictly less than κ , and its normalized stress entries strictly exceed σ_* .

Enclose each coefficient of $\mathbf{H}$ using the construction intervals, then enclose every entry of $\text{Id} - \mathbf{P}\mathbf{H}$. The sum of the entrywise absolute upper bounds in a row bounds that row’s absolute sum. Let ρ_{inv} be the largest resulting row bound and set

$$\varepsilon_{\text{inv}} = \frac{\rho_{\text{inv}} \|\mathbf{P}\|_{\infty}}{1 - \rho_{\text{inv}}}.$$

The sufficient certificate bounds are $\rho_{\text{inv}} < 10^{-10}$, $\|\mathbf{P}\|_{\infty} + \varepsilon_{\text{inv}} < \kappa$, and every entry of row $n_{\text{var}} + 1$ of $\mathbf{P}$ to exceed $\sigma_* + \varepsilon_{\text{inv}}$. Here rows are counted from 1, as in Proposition 5.5. Lemma 5.6 then proves both the actual inverse bound and the strict positive stress bound. The local computations verify all these comparisons over the full construction intervals.

Table 28 records the obtained bounds before they are weakened to the simpler constants used in the case proofs. Every entry is computed with rational arithmetic and rounded in its indicated direction. The stress column bounds every entry of row $n_{\text{var}} + 1$ of the actual inverse, by the error estimate just given.

Reference	$10^{12} \rho_{\text{inv}} \leq$	$\ \mathbf{H}^{-1}\ _{\infty} \leq$	Each $w_a \geq$
$\mathcal{P}_{16}$	.059	4.62041	.004679
$\mathcal{P}_{17}$	.062	7.96772	.004101
$\mathcal{Q}_1$	.092	2.55885	.015845
$\mathcal{Q}_2$	.097	2.18496	.015963
$\mathcal{Q}_3$	.093	2.49349	.015573
$\mathcal{Q}_4$	.091	2.06761	.010470
$\mathcal{Q}_5$	.089	2.26553	.010678
$\mathcal{Q}_6$	.070	2.97043	.015094
$\mathcal{C}$	.069	6.09602	.007930

Table 28: Outward-rounded rational certificate bounds. The residual and inverse columns are rounded upward; the stress column is rounded downward.

H Implementation and reproducibility

The computations use the coverage proof of Section 4 and the exact arithmetic of Appendix B. This appendix identifies the programs and data, explains which operations are trusted, and records the completed run. Instructions for executing the checks are in the README supplied with the source distribution.

H.1 Programs, certificates and records

The separate supplement `circle-packings-supplementary-materials.zip` contains the verification programs, rational angle tables, integer matrix arrays and completed verification output. All programs share one `supplementary-materials` directory, with subdirectories only for the five values of N . The supplied supplement is the computational archive for this paper. Its `README.txt` gives execution instructions, dependencies and successful completion conditions.

Table 29 maps every computational proposition to its programs. Paths in the table are relative to the `supplementary-materials` directory. A program used for more than one statement appears in more than one row so that each computation has an explicit correspondence. The auxiliary fifteen-circle derivative check repeats the rational comparisons proved in Appendix E.

Table 29: Computational propositions and their implementations.

Proposition or hand-checked calculation	Programs and input data
7.2: geometry for 15	<code>constants.py</code> ; <code>n15/pentagon_certificate.py</code>
7.3: inner counts for 15	<code>n15/inner_energy_certificate.py</code>
7.4: localization for 15	<code>n15/radial_integer.cpp</code> ; <code>n15/radial_integer_validate.py</code> ; <code>n15/radial_integer_angles.tsv</code>
Appendix E: derivative comparisons	<code>n15/additive_estimates.py</code>
8.1: construction for 16	<code>n16/certify_construction16.py</code>
8.2: geometry for 16	<code>constants.py</code> ; <code>n16/certify_geometry16.py</code>
8.3: inner counts for 16	<code>n16/energy16.py</code>
8.4: localization for 16	<code>n16/radial_integer.cpp</code> ; <code>n16/radial_integer_validate.py</code> ; <code>n16/radial_integer_angles.tsv</code>
8.5: positions for 16	<code>n16/coordinate_nearness.py</code>
8.6: local certificate for 16	<code>n16/certify_local16.py</code> ; <code>n16/inverse16.json</code>
8.7: original positions for 16	<code>n16/certify_construction16.py</code>
9.1: construction for 17	<code>n17/verify17_exact.py</code>
9.2: geometry for 17	<code>constants.py</code> ; <code>n17/validate_angles.py</code>
9.3: inner counts for 17	<code>n17/energy17.py</code>
9.4: exclusion and localization for 17	<code>n17/radial_integer.cpp</code> ; <code>n17/validate_angles.py</code> ; <code>n17/radial17_angles_6.tsv</code> and <code>radial17_angles_7.tsv</code> in the same directory
9.5: positions for 17	<code>n17/coordinate_nearness17.py</code>
9.6: local certificate for 17	<code>n17/certify_local17.py</code> ; <code>n17/inverse17.json</code>
10.1: geometry for 18	<code>constants.py</code>
10.2: inner counts for 18	<code>n18/energy18.py</code>
10.3: the six references	<code>n18/exact18.py</code>
10.4: matching for 18	<code>n18/radial_integer.cpp</code> ; <code>n18/validate_angles.py</code> ; <code>n18/radial18_angles_7.tsv</code> and <code>radial18_angles_8.tsv</code> in the same directory
10.5: six local certificates	<code>n18/certify_m10.py</code> ; <code>n18/certify_m11_limits.py</code> ; <code>n18/certify_local18.py</code> ; the six inverse arrays specified below
10.6: angle derivatives; Lemma 10.7	<code>n18/certify_bridges18.py</code>
11.1: construction for 20	<code>n20/verify20_exact.py</code>

Proposition or hand-checked calculation	Programs and input data
11.2: geometry for 20	<code>constants.py</code> ; <code>n20/validate_angles.py</code>
11.3: inner counts for 20	<code>n20/energy_integer.cpp</code> ; <code>n20/energy_integer_validate.py</code> ; <code>n20/energy20.py</code> ; <code>n20/energy20_angles.tsv</code>
11.4: matching for 20	<code>n20/radial_integer.cpp</code> ; <code>n20/validate_angles.py</code> ; <code>n20/radial20_angles.tsv</code>
11.5: positions for 20	<code>n20/coordinate_nearness20.py</code>
11.6: local certificate for 20	<code>n20/certify_local20.py</code> ; <code>n20/inverse20.json</code>
11.7: original positions for 20	<code>n20/verify20_exact.py</code>

In the eighteen-circle data, the names `m10_a`, `m10_b`, `m11_lim3`, `m11_lim6`, `m11_lim8` and `m11_core` denote $Q_1, \dots, Q_6$, in that order. The first five inverse arrays have names formed by prefixing `inverse_` and appending `.json` to those names; the sixth is `n18/inverse18.json`. All six are in `n18/`. Their exact dimensions, scales and coordinate ordering are specified in Appendix G.

The additional program `certificate_summary.py` recomputes the root and matrix summaries in Tables 13 and 28 using the same exact interval routines. It rounds each reported bound outward with integer division and compares the result with `certificate_summary.json`. These checks are included in the completed verification.

The driver `verify.py` validates the angle data before each integer subdivision and requires every computation to finish successfully. Its output records the exact subdivision counts in Table 20. The supplied record is `verification-output.txt`; `verification-results.json` gives the completion status, software versions and elapsed times. The recorded run completed in 490.1 seconds (about 8.2 minutes) with Python 3.12.14 and GCC 13.3.0 on Linux x86-64. The stages ran sequentially. Runtime depends on the computing environment.

The shared checks also recompute the composition counts, compare every printed count and position bound with the exact output, and reconstruct the reference coordinates and matrix bounds by a second rational interval calculation. They are `check_compositions.py`, `check_consistency.py`, and `check_reference_certificates.py`. The long root endpoints and the printed gap bounds for sixteen circles are checked by `n16/check_printed_bounds.py`.

H.2 Pseudocode for the finite checks

The procedures below use closed rational intervals. An exclusion discards a box only by a necessary inequality proved in Section 4. A match accepts the whole box only when all its retained intervals lie in one permitted target. Failure means that the computation has not established its conclusion.

Algorithm H.1 (Resolve one full cyclic order).

Input: a count vector $\mathbf{q}_{\text{out}}$ and its initial radial box. All pair-angle bounds used below have first been certified as in Appendix B.

Put the initial box on a stack.

while the stack is nonempty:

 Remove its last box $\mathbf{I}$.

 Make two passes through all inner pairs $i < j$, in increasing index order.

 If $r_i^+ + r_j^+ < 2$, exclude $\mathbf{I}$ and process the next box.

 Otherwise set $r_i^- \leftarrow \max(r_i^-, 2 - r_j^+)$ and

$r_j^- \leftarrow \max(r_j^-, 2 - r_i^+)$.

 For each pair, take the least certified angle bound at its four corners.

 If the sum of the direct-or-chain gap bounds exceeds Π^+ , exclude.

 Otherwise form the directed bounds d_{ij} of (4.3).

 For each intermediate index j_0 , update $d_{ij} \leftarrow \max(d_{ij}, d_{ij_0} + d_{j_0j})$.

 After each j_0 , a positive diagonal entry excludes the box.

 If a complete target match holds, accept $\mathbf{I}$ and process the next box.

 Otherwise select an interval $[r_i^-, r_i^+]$ for bisection.

 If its width is less than two grid units, report failure.

 Split at the floor midpoint of its integer endpoint numerators.

 Push the upper child, then the lower child, with the midpoint in both.

return success when the stack is empty.

The gap sum in Algorithm H.1 uses the larger of the direct inner-pair bound and the chain bound in Lemma 3.1. An exclusion immediately starts the next iteration. Success is required for every dihedral representative in Table 20; the initial common radial ranges make that reduction valid by Section 4.2.

For fifteen through eighteen circles, select a widest interval, breaking ties by its smallest index. For twenty, first select the last index with upper radius below 1, if one exists, and prefer a widest interval among the other indices. When their largest width is at most 1024 radial grid units, split the selected interval if it is wider. This rule changes only the order of subdivision.

The simple matches for fifteen, sixteen and seventeen circles compare the prescribed count vector and each full radial interval with the targets in the corresponding proposition. The two other cases need the following deletion rule. A target consists of a cyclic radius list and its following outer counts; any target intervals used in the check are contained in the intervals stated in the proposition.

Algorithm H.2 (Match after omitting one inner direction).

Input: a box, its inner cyclic list, following outer counts, and the permitted targets from Proposition 10.4 or 11.4.

for each inner index to omit:

For eighteen circles, require its entire radius interval to lie in $[0, 10^{-6}]$.

Keep the remaining inner indices in their cyclic order.

Between each pair of retained neighbors, sum the intervening outer counts.

for each permitted target and each orientation:

List its inner centers and outer centers in that cyclic orientation.

for each cyclic alignment with the retained inner list:

Compare all following outer counts.

Check containment of every retained radial interval in its target.

If both comparisons hold throughout, return a match.

return no match.

Reversing the full cyclic list reverses which gap follows a matched inner center. Equivalently, if inner index j is matched to target index t in reverse order, its following count is compared with the target count at $t - 1$, with cyclic indices. The code uses this index rule. Every original center remains in the exclusion tests of Algorithm H.1 before the match is considered.

For eighteen circles, the omitted center is retained as the match to O . The target for $\mathcal{Q}_6$ initially has following counts $(1, 2, 2, 2, 2, 2)$; after a successful match, delete the sole outer center in the first gap to obtain its seventeen-center subset. For twenty circles, the omitted inner center is the remaining twentieth center. Its radius is unrestricted by the target comparison.

Algorithm H.3 (Exclude an inner count).

Input: k and its initial radial box from Table 17. Set $m = N - k$.

Put the initial box on a stack.

while the stack is nonempty:

Remove its last box. Make two passes through the following tests:

Contract using $r_1 \leq r_i$ and $r_2 \leq r_k$.

Contract using $r_i + r_j \geq 2$ for every pair.

An empty interval excludes the box.

If the certified lower bound for $\sum_i f_b(r_i)$ exceeds
the certified upper bound for $\pi - m\eta$, exclude the box.

Compute the pair-angle bounds by the four-corner rule.

For fifteen through eighteen circles, test the consecutive pair-angle sum.

Apply directed angle closure to the k inner centers.

If an angle test excludes the box, process the next box.

Otherwise bisect a widest interval, breaking ties by the smallest index.

If subdivision is impossible, report failure; otherwise push both closed children.

return success when the stack is empty, with every terminal box excluded.

The inner-count computations have no target acceptance step. Their minimum-radius and reversal conventions cover all inner cyclic orders by (4.2). The twenty-circle calculation uses directed closure without a separate consecutive pair-angle sum. Its starting upper radius is rounded outward to denominator 2^{30} ; the monotonicity bound for f_b also holds on this slightly enlarged interval.

Algorithm H.4 (Certify a local-rigidity bound).

Input: exact reference-coordinate enclosures, the proved contact list, rational additional columns V , and a rational inverse proposal P . The desired positive bounds are κ and σ_* .

Form the signed rigidity matrix A from the contact linear forms.

Delete the aligned horizontal coordinate. Form $H = [A \mid \mathbf{1} \mid V]$.

Require H and P to be square matrices of the same size.

Enclose each entry of $I_d - PH$ by rational interval arithmetic.

Let ρ_{inv} be the largest sum of absolute-entry upper bounds in a row.

Require $\rho_{\text{inv}} < 1$ and put

$$\varepsilon_{\text{inv}} = \rho_{\text{inv}} \|P\|_{\infty} / (1 - \rho_{\text{inv}}).$$

Require $\|P\|_{\infty} + \varepsilon_{\text{inv}} < \kappa$.

In the row corresponding to the adjoined column of ones, require

every entry of P minus ε_{inv} to exceed σ_* .

return the certified threshold $\sigma_*/(8\kappa)$.

The implementation checks the stronger residual bound $\rho_{\text{inv}} < 10^{-10}$. Lemma 5.6 and Proposition 5.5 prove that Algorithm H.4 supplies the two hypotheses of Lemma 5.4. The exact contact identities established in the construction sections make the selected linear forms valid constraints; the residual calculation supplies their stress and inverse bound.

H.3 Exact arithmetic and the trusted base

The Python programs use standard-library integers, rational arithmetic and integer square roots. The construction and matrix intervals use outward-rounded decimal grids, of denominator 10^{80} for sixteen and eighteen circles and 10^{60} for seventeen and twenty. Other rational tests use fractions directly or the particular grids stated in their specifications. The shared module `exact_arithmetic.py` supplies repeated rational trigonometric and square-root bounds.

Floating-point calculations propose angle bounds or inverse arrays. An angle proposal becomes a premise only after the rational cosine comparison of Appendix B succeeds. Every stored angle-table entry is checked before use. Each inverse array is checked against the actual interval contact matrix by the residual test in Appendix G. The arithmetic correctness of the proof therefore rests on these exact checks, not on the accuracy of the proposing calculations.

The C++ subdivisions use signed 64-bit integers for radius and angle numerators and signed 128-bit intermediates where required. They use no floating-point proposal mode in the recorded verification. Radius numerators are smaller than $4 \cdot 2^{40}$; sums, midpoint calculations and the direct gap sums stay within the 64-bit range. Cross-products for containment in decimal target intervals are smaller than $4 \cdot 2^{40} \cdot 10^{15} < 2^{92}$, within the signed 128-bit range. During directed closure, two 64-bit entries are added in the wider type and checked before conversion back. The shared routine is in `integer_bounds.hpp`; the twenty-circle count exclusion performs the same explicit check. Missing angle entries, failed bounds, unresolved boxes and incorrect completion counts cause failure.

The trusted execution base is the Python implementation and its standard library, the C++17 compiler and standard library, the specified integer semantics, and the operating system and hardware that execute them. Assertions must be enabled throughout. The verification record identifies the versions used; the build and figure software is separate from the proof checks. Figures use approximate coordinates for illustration and supply no premise.

I Notation and terminology

All angles in the paper are in radians. A superscript star marks reference data; a superscript $-$ or $+$ on an endpoint marks a lower or upper bound. Bold lowercase letters denote plane vectors, except the columns explicitly identified in the linear systems. Sans-serif capitals denote operators or matrices, and calligraphic capitals denote sets, apart from the discriminant $\mathcal{D}$ and the total angular bound $\mathcal{L}$.

Point labels are local to a case section: $\mathbf{A}$, $\mathbf{Q}_j$ and other bold capitals denote that section's reference centers. During the construction they are functions of the trial radius b ; afterward they are evaluated at b_N . A case appendix explicitly adopts the same notation. In tables comparing cases, the column heading identifies the construction. Once a case fixes $b = b_N$, it writes $c = c(b)$, $\eta = \eta(b)$, $h = h_b$ and $f = f_b$. Indices i, j enumerate the centers or gaps of the list specified where they occur. An index read modulo a list's length is stated explicitly at that use.

I.1 The problem and geometric bounds

Symbol	Meaning	Definition
N, R_N	Number of unit disks and its least container radius	Section 1
R, b, b_N	Container radius; available radius $b = R - 1$ for centers; exact construction radius	(1.1), case constructions
$O, \mathbf{x}_i, \mathcal{X}$	Origin, a center, and a finite feasible center set	(1.1), Lemma 2.2
$\hat{\mathcal{X}}, \hat{\mathbf{x}}_i$	Projected set and its centers	Section 2.2
r, s, r_i, θ_i	Radial variables; the radial distance and chosen direction of center i	Section 2
$\mathbf{e}(\theta)$	Unit vector $(\cos \theta, \sin \theta)$	Theorem 1.1
$D(r, s; \theta)$	Squared distance for two radii and an angle	(2.1)
$c(b), k, m$	Projection cutoff; inner and outer counts	(2.4), Section 2.2
$g(r, s)$	Required smaller angular separation	(2.6)
$\chi_b(r)$	Cosine of $h_b(r)$ on its positive branch	(2.12)
$\theta_{\text{out}}, \theta_{\text{in}}$	Directions of the illustrated outer and inner centers	Figure 3
$\eta(b), h_b(r)$	Half the angle between two outer centers in pair contact; required angle from radius r to radius b	(2.7), (2.9)
$r_*(b), f_b(r)$	Threshold where $h_b = \eta$; assigned inner half-width	(2.10), (2.13)
b_-, b_+, c_+, r_*^-	Certified radius bounds used in the geometric tests	Lemma 2.6, (4.7)
d, ν, μ	Auxiliary cutoff $d = c_+ - 4/c_+$ and count parameters for its two radial layers	Lemmas 2.7, 2.8
τ_b, R_b	$\sqrt{b^2 - 1}$; rotation through $2\eta(b)$	(6.1)
$\mathbf{J}, \mathbf{S}, \mathbf{l}_\pm$	Quarter-turn, vertical-axis reflection, and the two radius-2 circle intersections	Section 6
λ_m, ω_m	Half the unused outer angle and half the maximum outer gap	(3.1)
$q_i, q'_i, \mathbf{q}_{\text{out}}$	Outer counts, their reversed list, and the full count vector	Lemma 3.1, Sections 4 and 9
δ_i, δ_i^*	Consecutive angular gap in the specified list, and its reference value	(3.4), (10.5)
$r_{\text{cap}}, \omega_m^+$	Proposed radial cap and upper bound on ω_m	(3.5)
$t_{\text{cap}}, s_{\text{cap}}$	Comparison variables in the radial-cap proof	Lemma 3.2
$t_b(\theta)$	Smaller radial solution of the distance-2 equation	(3.7)
$\ell_{\text{gap}}, u_{\text{gap}}, v_{\text{gap}}$	Radial constants for the one-gap enclosure	(3.8)
$\mathcal{V}_{\text{gap}}$	Five vertices defining the enclosing convex hull	(3.12)
$\mathbf{u}_{\text{gap}}, \mathbf{l}_\pm, \mathbf{v}_\pm$	The individual enclosure vertices	(3.12)

Symbol	Meaning	Definition
α_{sep}	Smaller angle between the two directions in the projection proof	Lemma 2.2
$\alpha, \mathbf{p}_{\pm}, F_{\text{gap}}$	Half-width of a chosen outer gap, its enlarged end-points, and the circle's left boundary function	Proof of Lemma 3.4
$\alpha_i, \beta_j, \mathbf{w}_i$	Convex weights and vertices in the hull-diameter proof	(3.17)

I.2 Intervals, local bounds and linear systems

Symbol	Meaning	Definition
s_i, n	Rational interval split point and size of a full cyclic center list	Section 4
$I_i, \mathbf{I}, r_i^-, r_i^+$	Radial interval, product box and its endpoint radii	(4.1)
$I, J, L(r', s'), L$	Two radial intervals, certified corner angle bounds and their minimum	Lemma 4.1
L_{ij}, d_{ij}, Π^+	Pair-angle lower bound, directed-angle lower bound and full-turn upper bound	(4.3)–(4.4)
$\eta_-, \eta_+, \underline{g}, Q_r$	Bounds on η , certified lower bound for g , and its cosine quotient with radius r fixed	(4.9) and its proof
$\mathcal{D}(r, s)$	Discriminant in the angle derivatives	(5.1)
F, t_0, t_1	Real function and interval endpoints in the calculus estimates	(5.5)
$M_{F,1}, M_{F,2}$	Absolute bounds for the first and second derivatives of F	(5.5), Lemma 5.2
e_F, J_F	Taylor error and auxiliary function in its proof	Lemma 5.2
$\mathbf{x}_i^*, r_i^*, \theta_i^*$	Reference center, radius and direction	Lemma 5.3
$\mathbf{x}_{\mathbf{p}}$	Unknown center matched to reference center $\mathbf{p}$	Case sections
$\varepsilon_r, \varepsilon_\theta, \varepsilon_{\text{pos}}$	Radial, angular and resulting position error bounds	(5.7), (5.8)
$n_{\text{ref}}, i_0, \mathbf{v}_i, U$	Reference size, aligned-center index, displacement vector and largest absolute coordinate change	(5.9)
C_{loc}	Positive constant in the motivating inequality $U \leq C_{\text{loc}} U^2$	Before Lemma 5.4
$x_{i,x}^*, x_{i,y}^*$	Cartesian coordinates of reference center i	Section 5.3
$y_{ij}, y_i^{\text{bd}}, y_a$	Pair, container and enumerated linearized constraints	(5.10), (5.12)
$n_{\text{con}}, \mathbf{y}_{\text{lin}}$	Constraint count and column of constraint values; prescribed column in the linear systems	(5.13), (5.17)
$w_a, \mathbf{w}_{\text{stress}}, \sigma_*, \kappa$	Normalized stress weights, their column, a strict lower bound for them and an inverse bound	(5.14)–(5.15)
$n_{\text{var}}, \mathbf{v}', \mathbf{A}$	Free-coordinate count, displacement column and signed rigidity matrix	(5.17)
$\mathbf{1}, \mathbf{V}, \mathbf{H}$	Column of ones, rational extra columns and square augmented matrix	(5.18)
$\mathbf{P}, \mathbf{E}, \mathbf{Id}$	Proposed inverse, residual matrix and identity matrix	(5.20)
$\rho_{\text{inv}}, \varepsilon_{\text{inv}}$	Residual norm bound and resulting inverse error bound	Lemma 5.6
$\ \cdot\ _\infty$	Maximum absolute vector entry or maximum absolute matrix row sum	(5.21)
$\mathbf{z}_{\text{lin}}, \mathbf{M}, \mathbf{N}$	Column variable and generic matrices in the norm and residual estimates	(5.21)

Symbol	Meaning	Definition
z_1, z_2, s_{aux}	Two free scalar variables and the auxiliary variable in the three-row example	(5.19)

I.3 Construction-specific and certificate data

Symbol	Meaning	Definition
$\mathcal{P}_N$	Exact N -center construction	Case construction sections
$\mathbf{A}, \mathbf{C}, \dots$	Reference-point labels local to the indicated case section	Case construction sections
F_{16}, F_{17}, F_{20}	Closing functions defining the three algebraic radii	(8.5), (9.6), (11.3)
b_N^-, b_N^+	Rational endpoints of a construction-root interval	Appendix A.1, (9.1), (11.1)
b_{16}^{base}	Rational common part of the long endpoints	Appendix A.1
φ, a, t, γ	$\pi/5$, pentagon circumradius, apothem and $\cos \varphi$	(7.1), Section 7.1
$\xi_i, \xi, \zeta, \varepsilon$	Proportional radial changes and their local upper bound 1/200 (fifteen-circle case)	(7.16), (7.17)
$\mathcal{L}$	Sum of the larger direct and contact-chain gap bounds	(7.14)
δ_j^{sq}	Three successive spacings in a square contact chain, $1 \leq j \leq 3$	Section 7
Φ, Ψ, χ	Scaled angle functions and $1/t$	(7.17), Lemma 7.5
F_{seg}, λ	Restriction of Φ to a radial segment and its parameter	(7.22)
X_j, Y_j	Cartesian coordinates of $\mathbf{Q}_j$	(8.1)
e, α_E, β_E	Squared radius of $\mathbf{E}$ and coefficients defining $\mathbf{H}$	(9.2)
J_x, J_y	Cartesian coordinates of $\mathbf{J}$ in the seventeen- or twenty-circle case	(9.4), (11.2)
t_i, t'_i	Rational targets for the seven inner radii, in the two orientations	(9.12) and following text
$\mathcal{Q}_j, \Omega$	Six reference sets and the dodecagon directions	Table 5, Section 10.3
n_2, n_c	Counts of radius-2 and cutoff-radius reference centers	(10.6)
a_0, a_2, a_3, a_6	Rational coefficients in the field $\mathbb{Q}(\sqrt{2}, \sqrt{3})$	Section 10.3
$\varepsilon, n_{\text{loss}}$	Local matching tolerance 10^{-6} (eighteen-circle case) and number of gaps that may decrease	(10.7), Lemma 10.7
$\mathcal{J}, \Delta_{\mathcal{J}}$	Consecutive block of gap indices and its total angle change	(10.10)
$\mathcal{C}, \mathcal{A}, \mathbf{z}$	Fixed nineteen-center set, feasible region for the remaining center, and that center	(1.3), Section 11.1
$\vartheta_{\mathbf{p}}, \ell_j, \Lambda$	Matched-center angle, pair bound in a worked-example table, and the first sixteen-circle chain sum	Appendix D
$\alpha^{\pm}, \beta^{\pm}$	Squared-radius bounds for the derivative estimates	(E.2)
$\mathcal{D}_{\text{in}}^-, \mathcal{D}_{\text{out}}^-$	Rational discriminant lower bounds	(E.3), (E.5)
$M_1, \dots, M_4$	Rational bounds for the terms of the second derivatives	Appendix E
I_2, I_c, I_b	Coarse enclosing intervals in the eighteen-circle derivative checks	Appendix F
$Q_{\text{grid}}, j_{\text{sqrt}}$	Rational-grid denominator and integer square-root bound	Appendix B
$x_{\text{int}}, y_{\text{int}}, \iota_{\pm}, j_{\pm}$	Real interval values and their rational endpoints	(B.1)

Symbol	Meaning	Definition
$z, n_{\text{deg}}, n_{\text{terms}}$	Real series argument, Taylor degree and arctangent term count	Appendix B
$P_{\cos,26}, P_{\cos,34}$	Cosine lower polynomials of the indicated degrees	(B.4)
α_M, β_M	The two arctangents in Machin's identity	(B.5) and its proof
Q_θ, Q_P, Q_V	Angle denominator, inverse-array scale and extra-column scale	(B.7), Appendix G
$n_{\text{split}}, n_{\text{leaf}}$	Numbers of split and terminal boxes in a subdivision tree	Appendix C

I.4 Terminology

A center's *radial distance* is its distance from O . A *separated* set has pairwise distances at least 2. A *feasible center set* also satisfies containment (1.1). After projection, an *inner center* has radius less than $c(b)$, and an *outer center* has radius b and originally had radius at least $c(b)$. A *boundary center* has radius b . An *interior reference center* has radius less than b , possibly equal to $c(b)$.

A *contact pair* has center distance exactly 2. A *container contact* means that a unit disk touches the container, equivalently that its center has norm b . A *contact chain* is a sequence of centers with successive members in pair contact. An *outer gap* is the closed angular interval between successive outer directions. A *cyclic order* lists directions around the origin, with a chosen ordering for ties; its *cyclic count vector* records the intervening outer counts. *Dihedral equivalence* identifies shifts and reversal of this vector, together with the corresponding relabeling.

A *radial box* is a product of closed radius intervals. An *excluded box* fails a necessary inequality throughout; an *accepted box* lies wholly inside a stated target match. A *terminal box* is one at which subdivision stops. A completed successful calculation resolves every terminal box by exclusion or acceptance. *Directed angle closure* adds valid lower bounds on differences of ordered angles.

A *linearized constraint* is the first-degree part of a squared-distance or containment constraint at the reference set. Its coefficients form the *signed rigidity matrix*. An *equilibrium stress* is a list of weights whose weighted linearized sum is identically zero; the stresses used here are strictly positive and normalized to sum to 1. *Infinitesimal rigidity* excludes nonzero feasible first-order changes after alignment. *Local rigidity* excludes all other feasible sets in a specified neighborhood after alignment. A *residual certificate* verifies an inverse estimate from $\text{Id} - \text{PH}$.

A *rattler* is a circle that can move while the other circles remain fixed. An *isometry* preserves plane distances. Congruence allows one isometry of the packing and container together; fixing O leaves rotations and reflections.

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